Optimal Fractal-Like Hierarchical Honeycombs

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Hexagonal honeycomb structures are known for their high strength and low weight. We construct a new class of fractal-appearing cellular metamaterials by replacing each three-edge vertex of a base hexagonal network with a smaller hexagon and iterating this process. The mechanical properties of the structure after different orders of the iteration are optimized. We find that the optimal structure (with highest in-plane stiffness for a given weight ratio) is self-similar but requires higher order hierarchy as the density vanishes. These results offer insights into how incorporating hierarchy in the material structure can create low-density metamaterials with desired properties and function.

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Hierarchically structured material systems are characterized by the existence of structure at different length scales and often exhibit superior mechanical properties such as enhanced stiffness [1,2], strength [2,3], toughness [4–6], and negative Poisson’s ratio [7–9]. They are used in many fields including polymers [10], composite structures [11–13], sandwich panel cores [14,15], and biomimetic systems [2,6,16]. Perhaps the simplest example of an object whose stiffness is increased by structure is the simple hexagonal honeycomb [17]: such objects are well known to have relatively high stiffness for their low density. Recent work has sought to improve the properties of such structures by hollowing out the elements and replacing them with repeating units [18]. Along these lines, we consider a new family of honeycomb structures with a hierarchical refinement scheme in which the structural hexagonal lattice is replaced by smaller hexagons. This process can be repeated to create honeycombs of higher hierarchical order (see Fig. 1). As well as being a natural way to generate hierarchy, a similar structure has previously been proposed as a natural one for a two-dimensional soap froth to take [19–21] and is reminiscent of micrographs of polymeric foam which suggest two levels of hierarchy [2]. Such cellular solids have previously been shown to have improved in-plane stiffness and strength compared to the corresponding regular honeycombs [1,2,22,23]. However, it is still unknown whether such structures can be systematically optimized, in particular by adjusting the number of hierarchies that are used. In this Letter, the optimal configuration of such hierarchical honeycombs in the sense of highest elastic modulus is determined for various structural densities using finite element simulation, scaling analysis, and experiments.

The structural organization (a set of real numbers γi) is defined by the ratio of the newly introduced hexagonal edge length (li) to previous hexagon edge length (li−1) where i varies from 2 to n (hierarchical order) (i.e., γi = li/li−1). For convenience, γ1 is defined as 2l1/l0 (see below). Some geometric constraints on the hierarchically introduced edges must be imposed to avoid overlapping with preexisting

FIG. 1 (color online). (a) Unit cell of regular (i.e., zeroth) to fourth order hierarchical honeycombs fabricated using 3D printing. The physical thickness of the structures is constant, tn = 2 mm, because of the limitations of the 3D printing. To maintain the structure density, therefore, the size of this unit cell increases as the order of the hierarchy increases. (b) Unit cell of the hierarchical honeycombs with regular structure (left) and with first order hierarchy (right). Here F is an arbitrary concentrated force and N1 and N2 are the reaction forces at the midline.
the effective elastic modulus at fixed relative density as a function of thickness ratio \( t_n \), which can be written based on structural organization parameters as

\[
0 \leq \gamma_n \leq 1, \quad \text{and} \quad \sum_{i=1}^{n} \prod_{j=1}^{i} \gamma_j \leq 1, \tag{1}
\]

which must hold for all hierarchical orders \((n \geq 1)\). For simplicity, we assume that the wall thickness of \( t_n \) is uniform within a given structure; the relative density of the structure compared to the material density \( \rho_s \), i.e., \( \bar{\rho} = \rho/\rho_s \), can be related to the length ratios \( \{\gamma_i\} \) and \( t_n/l_0 \) via

\[
\bar{\rho} = \frac{2}{\sqrt{3}} \left( 1 + \sum_{i=1}^{n} \prod_{j=1}^{i} \gamma_j \right) \frac{t_n}{l_0}. \tag{2}
\]

This relation is used to adjust the thickness \( t_n \) to maintain a fixed relative density \( \bar{\rho} \) as the number of hierarchies, and the values of \( \gamma_i \), are varied.

A hexagonal honeycomb network extending spatially to infinity has sixfold rotational symmetry. Classic symmetry arguments show that threefold symmetry is enough to guarantee an isotropic in-plane linear response for a two-dimensional solid [24]. The macroscopic in-plane elastic behavior of a hexagonal honeycomb structure is therefore isotropic and can be described by two elastic moduli, which we take to be the Young’s modulus \( E \) and Poisson’s ratio \( \nu \). In this Letter, we focus on characterizing the effective Young’s modulus of the structure \( E \), measured relative to the Young’s modulus of the basic honeycomb structure \( E_0 \). For numerical and analytical analysis, the far-field uniaxial stress in the vertical direction, \( \sigma_{yy} = (-2/3)F/l_0 \), was imposed to determine \( E \). Here \( F \) is an arbitrary concentrated force; the vertical stress is equivalent to applying a force \( F \) in the vertical direction at the midpoint of every oblique edge in the original (i.e., zero hierarchy) hexagons (refer to the Supplemental Material for details [25]). To carry out the analysis, the unit cell of the lattice [Fig. 1(b)] was selected to represent the loaded lattice structure. Each beam in the lattice can undergo stretching, shear, and bending. In the bending dominated regime, the elastic modulus of the first order hierarchical honeycomb can be written as (see the Supplemental Material [25])

\[
\frac{E_1}{E_0} = \frac{\sqrt{3}}{4} f(\gamma_1) \left( \frac{t_1}{l_0} \right)^3, \tag{3}
\]

where \( E_0/E_0^m = (4/\sqrt{3})(t_0/l_0)^3 \) is the elastic modulus of a regular honeycomb with the same density [17], \( E_0^m \) is the material elastic modulus, and \( f(\gamma_1) = \sqrt{3}/(0.75-1.7625\gamma_1 + 0.9\gamma_1^2 + 0.3625\gamma_1^3) \). The element thickness ratio \( t_1/l_0 \) can be eliminated using Eq. (2), giving the effective elastic modulus at fixed relative density as

\[
\frac{E_1}{E_0} = \frac{\sqrt{3}}{4(1+\gamma_1)^3} f(\gamma_1). \tag{4}
\]

For higher order hierarchies, a finite element analysis was implemented using MATLAB. This allowed us to systematically change the geometry of the hierarchical structure and, in particular, to find the geometry, i.e., the set of \( \{\gamma_i\} \), that maximizes the effective elastic modulus of the honeycomb at a given order of the hierarchy. Figure 2 shows the maximum effective elastic modulus, normalized by the elastic modulus of a regular honeycomb with the same density \( E_0 = 1.5\bar{\rho}E_0^m \), for different relative structural densities (i.e., different values of \( \bar{\rho} \)) as the order of the hierarchy changes. As can be seen from this figure, the maximum effective elastic modulus saturates above a certain number of hierarchical orders. (We note that since the lower order hierarchies are special case of higher orders, the curves in Fig. 2 never reach a local maximum but merely saturate.) For example, for \( 0.018 \leq \bar{\rho} \leq 0.026 \), the maximum modulus is achieved for hierarchical orders \( \geq 6 \). This feature is confirmed by experiments in which unit cells of hierarchical honeycombs with one to four hierarchies were fabricated using 3D printing, maintaining a constant relative density of 0.054 [see Fig. 1(a)]. The fabrication and mechanical testing are described in the Supplemental Material [25]. The experimentally measured effective modulus shows good agreement with that predicted by the numerical simulations, see the inset of Fig. 2.

Figure 2 also shows the behavior of hierarchical structures in which the shear and stretching energies are eliminated from the analysis (dashed curve), so that only the bending energy remains. As the number of hierarchies increases, the
effective elastic modulus of this “bending-only” structure increases without bound (as expected since the curvature increases without bound and hence so does the bending energy). We therefore see that at high orders of hierarchy, shear and stretching “soften” the structure: a balance that is crucial in determining the optimal structure. Figure 3 shows how the optimal structural organization, i.e., the set \( \{ \gamma_i \} \), evolves as the relative density changes. As can be seen from this figure, as \( \bar{\rho} \to 0 \), the values of \( \gamma \to 1/2 \) and \( n \) increases.

We now seek to understand these numerical results using a scaling analysis: we seek the maximum amplification of the effective elastic modulus, when replacing a scaling analysis: we seek the maximum amplification of the effective elastic modulus of the first order bending-only structure [right of Fig. 1(b)] is 1.598 times that of the regular honeycomb [left of Fig. 1(b)]. However, for the first hierarchical order, only three of the

![Image](https://example.com/image.png)

FIG. 3 (color online). Topology of the stiffest hierarchical honeycombs at different relative densities. The results show the values of \( \gamma \) corresponding to the optimum structure of the hierarchical honeycombs at different relative densities. Maximum achievable hierarchical order and selected topologies of the stiffest hierarchical honeycombs in the specified relative density range are also shown at the top.
The above scaling relations show that if the order of hierarchy increases, the bending-based modulus increases while the shear-based modulus decreases. We expect that the elastic modulus of the combined structure should be optimal when

\[ \frac{E}{E_0} = c_1 \bar{\rho}^{-0.46} + c_2, \]

where \( c_1 \approx 2.15 \) and \( c_2 \approx -3.19 \) can be found from numerical data. Figure 4(b) shows the maximum achievable elastic modulus as a function of relative density. The results of scaling analysis are in good agreement with finite element simulations, especially for small densities (\( R^2 = 0.93 \)). Replacing the value of \( n \) from Eq. (8) in the elastic modulus of Eq. (6) gives the maximum reachable elastic modulus at each density as

\[ E/E_0 = c_1 \bar{\rho}^{-0.46} + c_2, \]

FIG. 5 (color online). Elastic modulus range for different order of hierarchy \( n \) versus relative density. Dashed curve shows the limiting elastic modulus of the hierarchical honeycomb for specified relative density [Eq. (9)].

In summary, a new class of fractal-appearing cellular metamaterials is introduced. Our results show that the effective elastic modulus of the developed cellular material can be increased significantly by increasing the hierarchical order while preserving the structural density.

Although our focus has been on optimizing the modulus of the structure, our results also show that the effective modulus can be tuned by varying \( \bar{\rho} \) and \( n \). Figure 5 shows these achievable elastic moduli for \( n \leq 10 \); the upper bound of this range (dashed curve) shows the maximum achievable elastic modulus for different densities, and is equivalent to that shown in Fig. 4(b). As can be seen from this figure, increasing the hierarchical order while preserving the structural density can significantly increase the effective elastic modulus of the hierarchical structure. Similarly, the maximum achievable hierarchical level is increased by reducing the structural density.
order while preserving the structural density. The optimal hierarchical level is also shown to be increased by reducing the structural density. This particular case of hierarchical refinement can be seen as a promising realization of enhancing performance by adding structural hierarchy. Moreover, the current work provides insight into how incorporating hierarchy into the structural organization can play a substantial role in improving the properties and performance of materials and structural systems and introduces new avenues for development of novel metamaterials with tailorable properties.

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