SOME NUMERICAL METHODS TO COMPUTE THE EIGENVALUES OF A TIME-DELAY SYSTEM USING MATLAB

ELIAS JARLEBRING*

Abstract. Several properties of time-delay systems can determined from the set of solutions of the characteristic equation. We mention some ways to numerically determine parts of this solution set using the popular tool Matlab. In particular, we give illustrative Matlab code for some methods and discuss methods suitable for problems of very large dimension.

In the first or second chapter of many books related to delays, we are faced with a formal description of several qualitative properties of time-delay systems using the solution set of the characteristic equation of a linear delay-differential equation (DDE). For a system of DDEs with a single delay, this transcedental equation

$$\det(-sI + A_0 + A_1 e^{-\tau s}) = 0 \tag{1}$$

is known to have an infinite number of complex solutions which, in general, can not be expressed explicitly in terms of elementary functions. We mention some of the numerical methods to compute solutions of the characteristic equation which are available for the popular software package Matlab. Since the simplicity of the implementation of some methods seems to be not widely known, we illustrate the essential ideas of some methods with a couple of lines of Matlab code.

An explicit formula. For some special cases, the solutions of (1) can be expressed explicitly with formula containing the inverse of $x \mapsto xe^x$, known as the Lambert W function:

```
s=lambertw(k,tau*a1*exp(-a0*tau))/tau+a0
```
(2)

Here $A_0 = a_0, A_1 = a_1 \in \mathbb{C}$ and $k \in \mathbb{N}$ is the branch index. The impact of this exact analytic explicit formula should not be overestimated, as one of the motivations for the introduction of the elementary-like function Lambert W was indeed to be able to express the solutions of the scalar characteristic equation [Corless, Connet, Hare, Jeffrey and Knuth *On the Lambert W Function*, Adv. Comput. Math. 5:329-359, 1996]. Note that it is only directly applicable to scalar problems. However, the formula does generalize to non-scalar DDEs with simultaneously triangularizable system matrices A_0 and A_1. The generalization given in [E. Jarlebring and T. Damm *The Lambert W function and the spectrum of some multidimensional time-delay systems*, Automatica, 43(12):2124-2128, 2007] contains a matrix-version of Lambert W. Obviously, this class of DDEs is also very restricted.

Discretization of the PDE-representation. There are mature numerical algorithms for the problem of determining eigenvalues of a matrix. Two branches of methods in the literature exploit this by approximating the characteristic equation in such a way that approximate solutions of (1) can be computed from the eigenvalues of a matrix. These types of methods are either based on an approximation of the solution map or a discretization of the PDE-representation of the DDE. We describe these two types of methods in this and the next subsection and give the ideas in terms of Matlab code in (3) and (4).

It is widely known that a DDE can be rewritten as a hyperbolic partial differential equation (PDE) with interrelated boundary conditions. The operator corresponding

*TU Braunschweig, Inst. Comp. Math., 38106 Braunschweig, Germany, (e.jarlebring@tu-bs.de), http://www.icm.tu-bs.de/~jarlebri

to the boundary value is a (so-called) infinitesimal generator, and its spectrum coincides with the solutions of the characteristic equation. Hence, the eigenvalues of a matrix resulting from a sufficiently fine discretization of the PDE will approximate the solutions of the characteristic equation.

The approach based on the discretization of the PDE-representation was taken in [Bellen and Maset, *Numerical solution of constant coefficient linear delay differential equations as abstract Cauchy problems*, Numer. Math. 84:351-374, 2000]. Similar ideas were used in [Breda, Maset, Vermiglio, *Pseudospectral approximation of eigenvalues of derivative operators with non-local boundary conditions* Appl. Numer. Math., 27:318-331, 2006] by applying a discretization scheme based on a Chebyshev nodes. The method can be implemented in three lines of code:

```
N=10;  n=length(A0); % Discretization nodes N and size of DDE n
D=-cheb(N-1)*2/tau;                                                    (3)
eig([kron(D(1:N-1,:),eye(n));[A1,zeros(n,(N-2)*n), A0]])
```

The code above uses on the function `cheb.m` which returns a Chebyshev differentiation matrix. This function is given in [Trefethen, Spectral Methods in MATLAB, 2000] and is publicly available on the book's home page.

Discretization of the solution operator. The linear operator transforming the initial function segment to the solution segment at some time-point is referred to as the *solution operator*. The other branch of methods are based on a discretization of the solution operator. This idea is used in the software package DDE-BIFTOOL [Engelborghs, Luzyanina, Roose, *Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL*, ACM Trans. Math. Software 28:1-24, 2002], where the solution operator is approximated by a linear-multistep discretization. More recently, the solution operator has also been approximated with a Chebyshev differentiation matrix:

```
N=10;  n=length(A0); % Discretization nodes N and size of DDE n
DD=cheb(N-1)*2/tau;
DN=kron([DD(1:end-1,:);[zeros(1,N-1),1]],eye(n));
MA=kron([eye(N-1,N);zeros(1,N)],A0);                                   (4)
MB=[kron([eye(N-1,N)],A1);kron([1,zeros(1,N-1)],eye(n))];
(log(eig(MB,DN-MA))+k*2*pi*i)/tau  % branch k
```

This approach was taken in [Breda, *Numerical computation of characteristic roots for delay differential equations*, PhD thesis, 2004] and in a series of papers by Butcher, Ma, Bueler, Averina and Z. Szabo in the more general setting of DDEs with periodic coefficients. The first paper of this series of papers [Butcher, Ma, Bueler, Averina and Szabo, *Stability of linear time-periodic delay-differential equations via chebyshev polynomials*, Int. J. Numer. Methods Eng., 59:895-922, 2004] appeared in 2004 and the most recent is [Bueler, *Error bounds for approximate eigenvalues of periodic-coefficient linear delay differential equations*, SIAM J. Num. Analysis 59:2510-2536, 2007]. Apparently, the works of the group of Breda and the group of Bueler occurred independently and around the same point in time.

Nonlinear eigenvalue problem. Problems of the type $\det(M(s)) = 0$, where $M(s)$ is a parameter dependent matrix, are sometimes referred to as *nonlinear eigenvalue problems.* This very general class of problems is typically computationally difficult and no globally convergent numerical methods are available. There are however a number of numerical methods which can be adapted to the problem at hand. Many methods are listed in [Mehrmann and Voss, *Nonlinear Eigenvalue Problems: A Challenge for Modern Eigenvalue Methods*, GAMM Mitteilungen 27:121-152, 2004] and [Ruhe, *Algorithms for the nonlinear eigenvalue problem*, SIAM J. Numer. Anal.

10:674-689, 1973]. The Matlab code for several of these methods are publicly available online.

The characteristic equation (1) clearly belongs to this class of problems and any of the methods for the general nonlinear eigenvalue problems can be applied to (1). Similar to the case for the linear eigenvalue problems, so-called projection type methods have turned out to be efficient for very large nonlinear eigenvalue problem. This also seems to be the case for DDEs. Parts of the spectrum of DDEs of dimension 10^6 are computed using a projection method in [Jarlebring, *The spectrum of delay-differential equations: numerical methods, stability and perturbation*, PhD thesis, 2008].

Other methods. The characteristic equation is a root-finding problem, $f(s) = 0$. Even though many traditional algorithms for the root-finding problem, such as Newton-iteration, secant method and Halley-iteration have a high-order local convergence, they are not often used for (1) in practice. Fast local convergence is sufficient for many root-finding problems. This is however typically not the case for (1). In practice it is often desirable to find *all* solutions of (1) within some region in the complex plane, say the (shifted) open right half plane. For a stability analysis, missing one eigenvalue is not acceptable. For those problems, fast local convergence is not sufficient.

It should also be noted that the representation (1) is not numerically stable, e.g. $f(s) \approx 0$ does not always imply that the approximation s is good. This holds in particular for large problems.

However, for problems of small dimension, determining the intersections of the level sets have been successfully used [Vyhlídal, *Analysis and synthesis of time delay system spectrum*, Ph.D. thesis, 2003] to find a large number of solutions of (1).

Example. Consider the DDE

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} x(t-\tau),$$

appearing in the modelling of a neural network [Campbell, Edwards, van den Driessche, *Delayed coupling between two neural network loops*, SIAM J. Appl. Math., 65:316-335, 2004]. We illustrate the use of (3) by applying it to this DDE. The numerically computed spectrum for $\tau = 1$ is given in the figure below.