

Geometric partitioning and robust ad-hoc network design

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Abstract

We present fast approximation algorithms for the problem of dividing a given convex geographic region into smaller sub-regions so as to distribute the workloads of a set of vehicles. Our objective is to partition the region in such a fashion as to ensure that vehicles are capable of communicating with one another under limited communication radii. We consider variations of this problem in which sub-regions are constrained to have equal area or be convex, and as a side consequence, our approach yields a factor 1.99 approximation algorithm for the continuous k -centers problem on a convex polygon.

1 Introduction

Our problem can be motivated in the context of multi-vehicle coordination. Suppose that there are n vehicles that must provide service or surveillance to a convex region C . Further suppose that any two vehicles can communicate with each other if the distance between them is less than some given threshold radius r . In order to ensure that all vehicles be able to communicate with one another (possibly through intermediate vehicles), it is natural to desire that the vehicles be configured in such a way that the communication network between them be *connected*. At the same time, in order to simplify operations and ensure that the entire region is covered, a natural strategy is to divide the region C into n sub-regions R_1, \dots, R_n , such that each of the n vehicles is assigned to one of the n sub-regions.

Before stating our problem formally, we find it useful to introduce some notational conventions. Given a set of points $\mathbf{x} = \{x_1, \dots, x_n\} \subset C$ and a threshold radius r , we let $G_r(\mathbf{x})$ denote the graph whose nodes $\{1, \dots, n\}$ correspond to the points x_1, \dots, x_n , and whose edges (i, j) correspond to those pairs of points (i, j) such that $\|x_i - x_j\| \leq r$, where $\|\cdot\|$ denotes the Euclidean norm. Given a *partition* R_1, \dots, R_n of C (that is, a set of sub-regions R_i such that $\bigcup_{i=1}^n R_i = C$ and $R_i \cap R_j = \emptyset$ for all $i \neq j$), we let $\prod_{i=1}^n R_i$ denote the *Cartesian product* of those sub-regions, so that $\mathbf{x} = (x_1, \dots, x_n) \in \prod_{i=1}^n R_i$ if and only if $x_i \in R_i$ for each $i \in \{1, \dots, n\}$.

We now state our problem formally: given a convex planar region C with area A and an integer n , we are interested in partitioning C into sub-regions R_1, \dots, R_n in such a fashion that for any $\mathbf{x} = (x_1, \dots, x_n) \in \prod_{i=1}^n R_i$ the graph $G_r(\mathbf{x})$ is connected for some radius r . We would like to minimize r among all such partitions. We can write our optimization problem as

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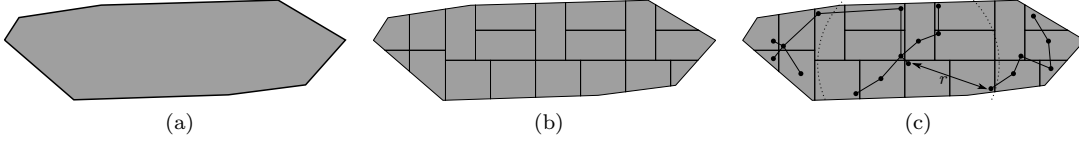


Figure 1: Input and output to problem (*). We are given a convex region C and an integer n (which is equal to 19 in this case), and our objective is to partition C into n sub-regions R_1, \dots, R_n ; we desire a partition whose *connectivity radius* is as small as possible. For the partition shown in 1b, the connectivity radius r is indicated in 1c; for any tuple $\mathbf{x} \in \prod_{i=1}^n R_i$, the graph $G_r(\mathbf{x})$ is connected.

$$\begin{aligned}
 & \underset{R_1, \dots, R_n, r}{\text{minimize}} && r && s.t. && (*) \\
 & G_r(\mathbf{x}) \text{ is connected} && \forall \mathbf{x} \in \prod_{i=1}^n R_i, \\
 & \bigcup_{i=1}^n R_i &= & C, \\
 & R_i \cap R_j &= & \emptyset.
 \end{aligned}$$

For a given partition, the minimum radius r that guarantees connectivity of all tuples $\mathbf{x} \in \prod_{i=1}^n R_i$ is called the *connectivity radius* associated with that partition. Inputs and outputs to problem (*) are shown in Figure 1. This formulation does not impose any additional constraints on the shapes or sizes of the sub-regions R_i ; it is of course sensible (from a practical standpoint) to add additional such requirements, such as convexity of the R_i 's.

Related work

This paper combines two related issues that commonly arise in multi-vehicle coordination, namely *geometric partitioning* and *ad-hoc network design*, and as such there exist two distinct bodies of work from which it stems. Geometric partitioning has emerged as a useful tool for allocating vehicles in a territory efficiently: for example, the papers [5, 6, 7, 8, 9, 12, 13] all describe various ways to balance the workloads of a fleet of vehicles (or facilities) by dividing the service region into sub-regions and localizing vehicles (or facilities) to those sub-regions. The paper [2] gives an algorithm for dividing a convex region into “fat” sub-regions of equal area, and in fact, Section 2 of this paper employs essentially the same algorithm for our problem. As we will later make clear, the constraint that all sub-regions have equal area can lead to situations in which the connectivity radius is quite high due to a “bottleneck” region at the periphery of the service region; this motivates our new algorithms in Sections 3 and 4. In addition, as a side consequence, our algorithm introduced in Section 3 can also be used to describe an approximation algorithm for the classical *k-centers problem* in a convex polygon with approximation factor 1.99; this is notable because it is known that there does not exist any approximation algorithm for the *k-centers problem* in a general metric space with a constant less than 2 unless P=NP; see for example [10].

The model for connectivity in our paper (namely, that two vehicles can communicate with each other if the distance between them is below the threshold r) is a common one and there exists a large

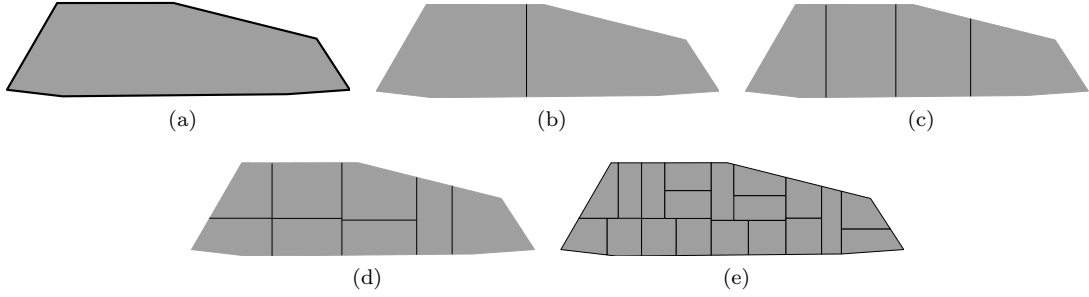


Figure 2: Input and output to Algorithm 1 for the case where we desire $n = 19$ sub-regions of equal area. In 2b, we use a vertical line to divide the region into two pieces whose areas are $9/19$ and $10/19$ of the original area. In 2c, we use vertical lines to further subdivide these two pieces into four pieces whose areas are $4/19$, $5/19$, $5/19$, and $5/19$ of the original area (from left to right). In 2d, we use vertical and horizontal lines to divide these pieces into even more pieces whose areas are all either $2/19$ or $3/19$ of the original area. Figure 2e shows the output of Algorithm 1, which consists of $n = 19$ sub-regions of equal area.

body of work dealing with various problems under such an assumption [1, 15, 16, 18, 19]. Generally speaking, the problems of interest often involve *decentralized* approaches to solving combinatorial problems along such a network, such as finding a connected dominating set, a minimal covering of the service region, or the shortest path between a pair of points.

Notational Conventions

Throughout this paper we use the following notational conventions: $\text{Area}(D)$ denotes the area of a region D . The *width* and *height* of a region are defined as the width and height of the minimum-area axis-aligned bounding rectangle of D and are denoted by $\text{width}(D)$ and $\text{height}(D)$. The *aspect ratio* of a rectangle R is the ratio of the length of the longer side of R to the the length of the shorter side of R and is written as $\text{AR}(R)$. The approximation ratio of each algorithm, i.e. the ratio of the upper bound to lower bound on the connectivity radius r , will be denoted Rat .

2 Imposing equal area and convexity

We begin by considering problem $(*)$ where we impose an additional constraint that all sub-regions R_i have equal area and be convex. The algorithm we use is due to [2] (which considers the closely related problem of partitioning C into “fat” sub-regions). This algorithm is quite simple: we rotate C so that its diameter is horizontal, and we then recursively divide C with either a horizontal line or a vertical line, depending on which of the two directions results in a “fatter” shape. This is described formally in Algorithm 1 and illustrated in Figure 2; we make a minor modification to the algorithm in order to improve our bound by applying a new Lemma 4. We find it helpful to introduce two lower bounds to our problem stated below:

Theorem 1. *Let C be an input to problem $(*)$ with an additional constraint that all sub-regions be convex and have equal area. Then $r^* \geq \sqrt{2A/\pi n}$.*

Input: A convex polygon C and an integer n .

Output: A partition of C into n convex sub-regions, each having area $\text{Area}(C)/n$.

Note: In the very first call of this algorithm – i.e. not the recursive calls – the input region C should be oriented so that its diameter is horizontal.

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if  $n = 1$  then
  | return  $C$ ;
else
  | Set  $n_1 = \lfloor n/2 \rfloor$  and  $n_2 = \lceil n/2 \rceil$ ;
  | Let  $w$  denote the width of  $C$  and  $h$  the height;
  | if  $w \geq h$  then
  |   | With a vertical line, divide  $C$  into two pieces  $R'_1$  and  $R'_2$  with area  $\frac{n_1}{n} \cdot \text{Area}(C)$  on the right and
  |   |  $\frac{n_2}{n} \cdot \text{Area}(C)$  on the left. Let  $w' = \max\{\text{width}(R'_1), \text{width}(R'_2)\}$ ;
  |   | With a vertical line, divide  $C$  into two pieces  $R''_1$  and  $R''_2$  with area  $\frac{n_1}{n} \cdot \text{Area}(C)$  on the left and
  |   |  $\frac{n_2}{n} \cdot \text{Area}(C)$  on the right. Let  $w'' = \max\{\text{width}(R''_1), \text{width}(R''_2)\}$ ;
  |   | if  $w' \leq w''$  then
  |   |   | Set  $R_1 = R'_1$  and  $R_2 = R'_2$ ;
  |   | else
  |   |   | Set  $R_1 = R''_1$  and  $R_2 = R''_2$ ;
  |   | end
  | else
  |   | With a horizontal line, divide  $R$  into two pieces  $R'_1$  and  $R'_2$  with area  $\frac{n_1}{n} \cdot \text{Area}(C)$  on the top and
  |   |  $\frac{n_2}{n} \cdot \text{Area}(C)$  on the bottom. Let  $h' = \max\{\text{height}(R'_1), \text{height}(R'_2)\}$ ;
  |   | With a horizontal line, divide  $R$  into two pieces  $R''_1$  and  $R''_2$  with area  $\frac{n_1}{n} \cdot \text{Area}(C)$  on the bottom
  |   | and  $\frac{n_2}{n} \cdot \text{Area}(C)$  on the top. Let  $h'' = \max\{\text{height}(R''_1), \text{height}(R''_2)\}$ ;
  |   | if  $h' \leq h''$  then
  |   |   | Set  $R_1 = R'_1$  and  $R_2 = R'_2$ ;
  |   | else
  |   |   | Set  $R_1 = R''_1$  and  $R_2 = R''_2$ ;
  |   | end
  | end
  | return  $\text{EqualAreaPartition}(R_1, n_1) \cup \text{EqualAreaPartition}(R_2, n_2)$ ;
end

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Algorithm 1: Algorithm $\text{EqualAreaPartition}(C, n)$ is due to [2]; it takes as input a convex polygon C and a positive integer n . We have made a minor modification to that of [2] in that we select a partition by comparing w' and w'' (or h' and h'').

Proof. Suppose that R_1, \dots, R_n is a partition of C into convex pieces of equal area and that r is sufficiently large that the graph $G_r(\mathbf{x})$ is connected for all $\mathbf{x} \in \prod_{i=1}^n R_i$. For any sub-region R_i and any point $x_i \in R_i$, it must be the case that there exists some *other* sub-region R_j such that R_j is entirely contained within a ball B_i of radius r centered at x_i (if this were not the case, then $G_r(\mathbf{x})$ would not be connected because we could isolate point x_i from the rest of the nodes). However, by the convexity assumption, R_i and R_j must be linearly separable, say by line ℓ . We therefore find that R_j must lie on one side of ℓ (say in half-plane H_ℓ), with x_i lying on the other side, and therefore R_j must be contained in $B_i \cap H_\ell$. Since $\text{Area}(B_i \cap H_\ell) \leq \pi r^2/2$ and $\text{Area}(R_j) = A/n$ by the equal-area assumption, we have

$$\begin{aligned} \frac{\pi r^2}{2} &\geq \text{Area}(B_i \cap H_\ell) \geq \text{Area}(R_j) = A/n \\ \implies r &\geq \sqrt{\frac{2A}{\pi n}} \end{aligned}$$

as desired. \square

Theorem 2. *Let C be an input to problem (*) with an additional constraint that all sub-regions be convex and have equal area and assume that C is oriented so that its diameter is horizontal. Let H_1 denote a vertically oriented half-space that cuts off exactly $\text{Area}(C)/n$ of C on the left and let H_2 denote a vertically oriented half-space that cuts off exactly $\text{Area}(C)/n$ of C on the right. Let w_1 and w_2 denote the widths of $C \cap H_1$ and $C \cap H_2$ respectively. Then $r^* \geq \max\{w_1, w_2\}$.*

Proof. Assume without loss of generality that $w_1 \geq w_2$. Consider a convex equal-area partition R_1, \dots, R_n and assume without loss of generality that the leftmost point of C , which we will call x_1 , is contained in R_1 . Since R_1 is convex, it is easy to see that $\text{Area}(R_1 \cap H_1) > 0$. We now claim that for any other region R_i , it must be possible to select a point $x_i \in R_i$ such that $\|x_1 - x_i\| \geq w_1$, which will complete the proof. This is straightforward: if there existed a region R_i such that this claim did not hold, then R_i would have to lie entirely to the left of H_1 (i.e. $R_i = R_i \cap H_1$). This is impossible because then

$$\text{Area}(C \cap H_1) \geq \text{Area}(R_1 \cap H_1) + \text{Area}(R_i \cap H_1) = \text{Area}(R_1 \cap H_1) + \text{Area}(R_i) > \text{Area}(C)/n$$

which contradicts our construction of H_1 . \square

2.1 Analysis of Algorithm 1

We will now show that Algorithm 1 solves (*) (subject to equal-area and convexity constraints) within a factor of 7.31. We shall make use of several pre-existing results from [2], which will be cited when appropriate.

Claim 3. Let $f(\cdot) : [0, 1] \rightarrow \mathbb{R}^+$ be a concave function such that $\int_0^1 f(t) dt = 1$. Then $\int_0^{1/4} f(t) dt < 1/2$ and $\int_{1/4}^{3/4} f(t) dt \geq 1/2$.

Proof. Let $h = f(1/4)$. As evident by Figure 3a, we see that certainly $\int_{1/4}^1 f(x) dx \geq 3h/8$, and therefore $\int_0^{1/4} f(t) dt \leq 1 - 3h/8$. It is also evident from Figure 3a that $\int_0^{1/4} f(t) dt \leq 7h/24$. Combining these two upper bounds we see that $\min\{1 - 3h/8, 7h/24\} \leq 7/16 < 1/2$ for all $h > 0$, which completes the proof of the first claim.

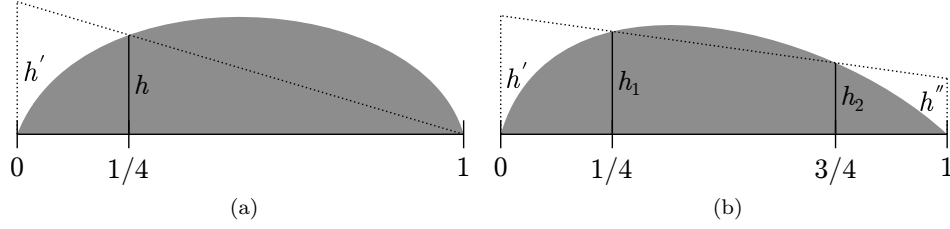


Figure 3: In 3a, we see that the area of the right triangle defined by points $(1/4, 0)$, $(1, 0)$, and $(1/4, h)$ is $3h/8$ and the area of the trapezoid defined by points $(0, 0)$, $(1/4, 0)$, $(1/4, h)$, and $(0, h')$ is $7h/24$. In 3b, the area of the trapezoid defined by points $(0, 0)$, $(1/4, 0)$, $(1/4, h_1)$, and $(0, h')$ is $(5h_1 - h_2)/16$, the area of the trapezoid defined by points $(1/4, 0)$, $(3/4, 0)$, $(3/4, h_2)$, and $(1/4, h_1)$ is $(h_1 + h_2)/4$, and the area of the trapezoid defined by points $(3/4, 0)$, $(1, 0)$, $(1, h'')$, and $(3/4, h_2)$ is $(5h_2 - h_1)/16$.

The second claim is similar; let $h_1 = f(1/4)$ and let $h_2 = f(3/4)$. Figure 3b shows that $\int_0^{1/4} f(t) dt + \int_{3/4}^1 f(t) dt \leq (h_1 + h_2)/4$, and therefore $\int_{1/4}^{3/4} f(t) dt \geq 1 - (h_1 + h_2)/4$. It is also evident from Figure 3b that $\int_{1/4}^{3/4} f(t) dt \geq (h_1 + h_2)/4$. Combining these upper bounds we see that $\max\{1 - (h_1 + h_2)/4, (h_1 + h_2)/4\} \geq 1/2$ for all $h_1, h_2 > 0$, which completes the proof of the second claim. \square

Lemma 4. *Let C be a convex polygon of width w . Let ℓ denote a vertical line that cuts off a fraction α of the area of C to the left and $1 - \alpha$ to the right, and let ℓ' denote a vertical line that cuts off $1 - \alpha$ to the left and α to the right. Let C_1 and C_2 denote the left and right pieces induced by ℓ and let C'_1 and C'_2 denote the left and right pieces induced by ℓ' . Then if $\alpha \in [1/3, 2/3]$, it must be the case that either*

$$\text{width}(C_1), \text{width}(C_2) \in [w/4, 3w/4]$$

or

$$\text{width}(C'_1), \text{width}(C'_2) \in [w/4, 3w/4].$$

Proof. Assume without loss of generality that $w = 1$ and that $\alpha \leq 1/2$. First, we “shift” C onto the horizontal axis, so that C can equivalently be represented as a concave function $f : [0, 1] \rightarrow \mathbb{R}^+$ as shown in Figure 4. We can also assume that $\int_0^1 f(t) dt = 1$ since we can scale $f(\cdot)$ arbitrarily in the vertical direction, i.e. replace $f(\cdot)$ by $af(\cdot)$ for some positive scalar a . It will therefore suffice to show that, for any such concave function, if we have

$$\int_0^c f(t) dt = \alpha \text{ and } \int_d^1 f(t) dt = \alpha,$$

then either $c \in [1/4, 3/4]$ or $d \in [1/4, 3/4]$. This follows from Claim 3. In particular, we merely have to rule out the two possibilities that $c, d < 1/4$ or that $c < 1/4$ and $d > 3/4$ (the case $c, d > 3/4$ is taken care of by symmetry).

1. Suppose for a contradiction that $c, d < 1/4$. Then by increasing α , we see that c must move to the right and d must move to the left, and therefore when we set $\alpha = 1/2$ we find that

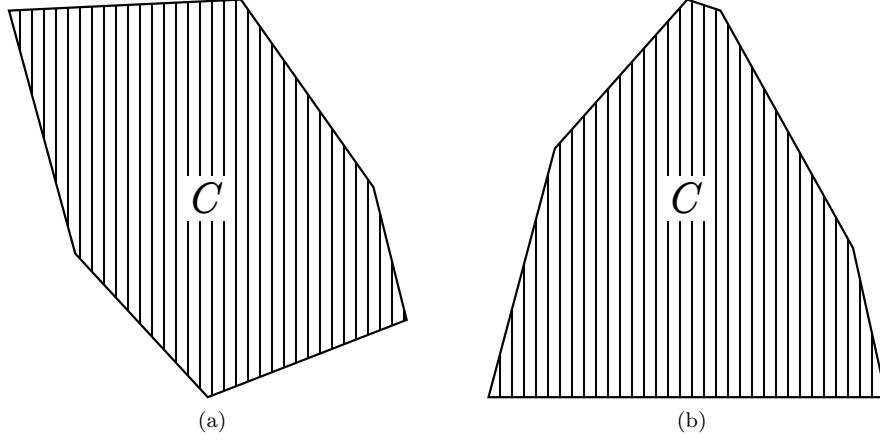


Figure 4: “Shifting” the polygon C onto the horizontal axis.

$c = d < 1/4$. Thus we have $\int_0^{1/4} f(x) dx > 1/2$, a contradiction of the first statement of Claim 3.

2. Suppose for a contradiction that $c < 1/4$ and $d > 3/4$. This implies that $\int_0^{1/4} f(x) dx > \alpha$ and $\int_{3/4}^1 f(x) dx > \alpha$. This would then imply that $\int_{1/4}^{3/4} f(x) dx < 1/3$, a contradiction of the second statement of Claim 3. This completes the proof.

□

Lemma 5. *For any sub-region R_i that is output by Algorithm 1, we have $\text{Area}(R_i) \geq \text{width}(R_i) \cdot \text{length}(R_i)/2$.*

Proof. This is Corollary 4.2 of [2] (which actually proves a stronger result, namely that the above also holds for any *intermediate* sub-region obtained throughout the execution of Algorithm 1, not just the final output R_1, \dots, R_n). □

Lemma 6. *Suppose that R_i is a sub-region output from Algorithm 1 such that $\text{AR}(R_i) > 4$. Then all cuts leading to R_i were vertical.*

Proof. This is essentially Corollary 4.6 of [2], which deals with the special case where $n = 2^k$ so that Algorithm 1 always divides sub-regions in half. By applying Lemma 4, we are able to generalize this result to arbitrary n (the paper [2] does indeed consider partitioning for general n , and their results hold for the case where $\text{AR}(R_i) > 6$). □

Lemma 7. *Suppose that R_i is a sub-region output from Algorithm 1 such that $\text{AR}(R_i) > 4$. Then there exists a unique leftmost sub-region or a unique rightmost sub-region R_{i^*} such that $\text{AR}(R_{i^*}) \geq \text{AR}(R_j)$ for all sub-regions R_j .*

Proof. This follows from the same proof as Claim 4.8 of [2]. □

Combining the above lemmas, we can now determine the overall approximation guarantee of Algorithm 1:

Theorem 8. *Algorithm 1 is a 7.31 approximation algorithm for problem (*) subject to the additional constraint that all sub-regions R_i be convex and have equal area. Its running time is $\mathcal{O}((m+n) \log n)$, where m is the number of edges of the input region.*

Proof. The running time of Algorithm 1 is given in Section 4.1 of [2]. The approximation ratio depends on the maximum aspect ratio of any of the sub-regions, $\max_i \text{AR}(R_i)$:

- If $\max_i \text{AR}(R_i) > 4$, then we apply Lemma 7 to consider the region R_{i^*} whose aspect ratio is as large as possible (which must be the leftmost or rightmost sub-region); let $z = \text{AR}(R_{i^*})$. It is then easy to verify that $\text{diam}(R_i) \leq \sqrt{2A(z + 1/z)/n}$ for all i (this is just the diagonal length of a rectangle with aspect ratio z and area $2A$; the “ $2A$ ” term arises apropos of Lemma 5). It also follows immediately that the radius of connectivity for our problem is at most twice that, i.e. that for any tuple $\mathbf{x} \in \prod_{i=1}^n R_i$, the graph $G_r(\mathbf{x})$ is connected, with $r = 2\sqrt{2A(z + 1/z)/n}$. Since $z > 4$ and R_{i^*} is either the leftmost or rightmost sub-region, we see that Theorem 2 gives us a lower bound that is simply the width of R_{i^*} , which is at least $\sqrt{Az/n}$. We therefore find that

$$\text{Rat} \leq \frac{\text{UB}}{\text{LB}} = \frac{2\sqrt{2A(z + 1/z)/n}}{\sqrt{Az/n}} = 2\sqrt{2(1 + 1/z^2)} < \sqrt{17/2} \approx 2.9155$$

since $z > 4$.

- If $\max_i \text{AR}(R_i) \leq 4$, then we must have $\text{diam}(R_i) \leq \sqrt{17A/2n}$ (this is, again, the diagonal length of a rectangle with aspect ratio 4 and area $2A$; the “ $2A$ ” term arises apropos of Lemma 5). It follows immediately that the radius of connectivity for the output of our algorithm is at most twice that, i.e. for any tuple $\mathbf{x} \in \prod_{i=1}^n R_i$, the graph $G_r(\mathbf{x})$ is connected, with $r = \sqrt{34A/n}$. Applying the lower bound of Theorem 1, we find that

$$\text{Rat} \leq \frac{\text{UB}}{\text{LB}} = \frac{\sqrt{34A/n}}{\sqrt{2A/\pi n}} = \sqrt{17\pi} < 7.31.$$

This completes the proof. \square

One of the practical drawbacks to Algorithm 1 is the interaction between the objective of minimizing the connectivity radius (which is in a sense a *maximum* distance taken over all sub-regions R_i) and the constraint that all sub-regions have equal area. Consider for example Figure 5, in which the input region is a right triangle with area 1 whose leftmost angle is $\theta = 10^\circ$. We therefore find that, for sufficiently large n , it will always be the case that $r^* \geq \sqrt{\frac{2}{\theta \cdot \frac{\pi \text{ rad}}{180^\circ} \cdot n}} = 6/\sqrt{\pi n}$, and one can plainly see that leftmost region acts as a bottleneck for the problem because all other sub-regions are much closer to their neighbors. Thus, it is natural to consider alternative problem formulations in which we relax the equal-area constraint (or the convexity constraint). We will next consider the fully unconstrained version of (*), which naturally brings other issues of its own (and which we will subsequently rectify).

3 The unconstrained version of (*)

We began the preceding section by introducing relevant lower bounds for (*) when convexity and equal area were imposed. We will do the same for this section, where we remove these restrictions:

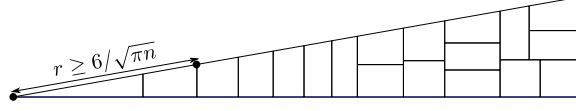


Figure 5: When the input region C is a right triangle with area 1 whose leftmost angle is $\theta = 10^\circ$, the leftmost sub-region always acts as a bottleneck for the overall connectivity radius.

Theorem 9. *The optimal solution r^* to problem (*) must satisfy*

$$r^* \geq \sqrt{\frac{A}{\pi + (n-1)(\pi/3 + \sqrt{3}/2)}}. \quad (1)$$

In order to prove Theorem 9, we require two simple lemmas:

Lemma 10. *For any partition R_1, \dots, R_n of C , there exists an n -tuple of points $\mathbf{x} \in \prod_{i=1}^n R_i$ and an index i^* such that $\|x_{i^*} - x_j\| \geq \sqrt{A/\pi n}$ for all $j \neq i^*$.*

Proof. Given any partition R_1, \dots, R_n , select the n -tuple \mathbf{x} arbitrarily and center an open ball B_i of radius $\sqrt{A/\pi n}$ at each element x_i . If there exists an index \bar{i} such that $B_{\bar{i}}$ is disjoint from all other balls, then clearly $i^* = \bar{i}$ and we are done. If no such element exists, then $\text{Area}(\bigcup_i B_i) < \sum_i \text{Area}(B_i) = A$ and therefore the balls B_i do not cover C . We can therefore select a point $x^* \in C \setminus \bigcup_i B_i$, so that $\|x^* - x_i\| \geq \sqrt{A/\pi n}$ for all i . Since R_1, \dots, R_n is a partition of C we know that $x^* \in R_{i^*}$ for some sub-region R_{i^*} . Setting $x_{i^*} = x^*$ completes the proof. \square

Proof of Theorem 9. Suppose that R_1, \dots, R_n is a partition of C and that r is sufficiently large such that the graph $G_r(\mathbf{x})$ is connected for all $\mathbf{x} \in \prod_{i=1}^n R_i$. Select any n -tuple $\mathbf{x} \in \prod_{i=1}^n R_i$ arbitrarily and center a ball of radius r at each element x_i . Following the same reasoning as the proof of Lemma 10, we see that the balls B_i must cover the region C . Furthermore, since $G_r(\mathbf{x})$ is connected, we know that $G_r(\mathbf{x})$ must have a spanning tree T . For notational simplicity, suppose without loss of generality that $1, \dots, n$ is a *pre-order traversal* [17] of T , so that for any index k , the subgraph of T associated with nodes $1, \dots, k$ is also a tree (and therefore connected). For any k , we will let T_k denote this subtree (so that $T = T_n$). Consider the leaf node $n \in G_r(\mathbf{x})$ of T and the portion of C that is *uniquely* associated with x_n , i.e. $(B_n \setminus \bigcup_{j=1}^{n-1} B_j) \cap C$. Since node n is within a distance r of at least one other node j^* , we see that portion uniquely associated with x_n is at most $(\pi/3 + \sqrt{3}/2)r^2$ (due to a direct computation of the area between two disks, i.e. a circular lens), whence

$$\text{Area} \left(\left(B_n \setminus \bigcup_{j=1}^{n-1} B_j \right) \cap C \right) \leq \text{Area} \left(B_n \setminus \bigcup_{j=1}^{n-1} B_j \right) \leq \text{Area}(B_n \setminus B_{j^*}) \leq (\pi/3 + \sqrt{3}/2)r^2.$$

Deleting node n (i.e. point x_n) and the region $B_n \setminus \bigcup_{j=1}^{n-1} B_j$, we now have a smaller region C' containing points x_1, \dots, x_{n-1} which are connected by the edges of T_{n-1} . It again must follow that the portion of C' that is uniquely associated with leaf node $n-1$ must satisfy

$$\text{Area} \left(\left(B_{n-1} \setminus \bigcup_{j=1}^{n-2} B_j \right) \cap C \right) \leq \text{Area} \left(B_{n-1} \setminus \bigcup_{j=1}^{n-2} B_j \right) \leq (\pi/3 + \sqrt{3}/2)r^2.$$

We can apply this process iteratively to find that, for each of the nodes $n, n-1, \dots, 2$, the area uniquely associated with that node must be at most $(\pi/3 + \sqrt{3}/2)r^2$. Our proof is complete by observing that, after the $n-1$ leaf deletions are completed, we have a single remaining point x_1 , which must obviously satisfy $\text{Area}(B_1) \leq \pi r^2$. Summing these together, we see that

$$A = \text{Area}(C) = \sum_{i=0}^{n-2} \text{Area} \left(\left(B_{n-i} \setminus \bigcup_{j=1}^{n-i-1} B_j \right) \cap C \right) + \text{Area}(B_1 \cap C) \leq \pi r^2 + (n-1)(\pi/3 + \sqrt{3}/2)r^2$$

from which (1) follows. \square

Note that Theorem 9 does not take the *shape* of C into account; it depends only on the area A . We find it necessary to introduce a second lower bound that applies when C is long and skinny:

Theorem 11. *The optimal solution r^* to problem (*) must satisfy*

$$r^* \geq d/n, \quad (2)$$

where d is the diameter of C .

Proof. This is similar in spirit to the proof of Theorem 9, restricted to one dimension. Again, suppose that R_1, \dots, R_n is a partition of C and that r is sufficiently large such that the graph $G_r(\mathbf{x})$ is connected for all $\mathbf{x} \in \prod_{i=1}^n R_i$. Select any n -tuple $\mathbf{x} \in \prod_{i=1}^n R_i$ arbitrarily and center a ball of radius r at each element x_i with . We again see that the balls B_i must cover the region C , and in particular, the balls B_i must cover the longest line segment s in C (whose length is by definition the diameter d of C). Project each point x_i onto s , obtaining a new n -tuple $\mathbf{x}' = (x'_1, x'_2, \dots, x'_n)$. Similarly, project each ball B_i onto s , obtaining line segments s'_1, \dots, s'_n of length at most $2r$. Since convex projection is a *nonexpansive* mapping (see e.g. [20]), we know that the graph $G_r(\mathbf{x}')$ must also be connected.

Assume without loss of generality that s is aligned with the horizontal axis and that the points (x'_1, \dots, x'_n) are sorted in order from left to right. Further assume that x'_n is the rightmost endpoint of s (which we are free to do because the initial tuple \mathbf{x} was chosen arbitrarily anyway), which implies that $\text{length}(s'_n) \leq r$. As in the proof of Theorem 9, consider the portion of s that is uniquely associated with s'_1 , i.e. $s'_1 \setminus \bigcup_{j=2}^n s'_j$. Since we know that $\|x'_1 - x'_2\| \leq r$ by connectivity of $G_r(\mathbf{x}')$, we find that

$$\text{length} \left(s'_1 \setminus \bigcup_{j=2}^n s'_j \right) \leq r$$

and, deleting x'_1 and s'_1 , we find that

$$\text{length} \left(s'_2 \setminus \bigcup_{j=3}^n s'_j \right) \leq r$$

and so on and so forth. We ultimately conclude that

$$d = \text{length}(s) = \sum_{i=1}^{n-1} \text{length} \left(s'_i \setminus \bigcup_{j=i+1}^n s'_j \right) + \text{length}(s'_n) \leq nr$$

from which the desired result follows. \square

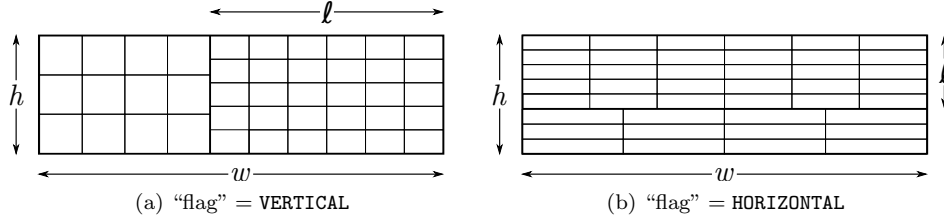


Figure 6: The output of Algorithm 2 where $(p_1, q_1, p_2, q_2) = (4, 3, 6, 5)$ and ℓ is as indicated. We subdivide the rectangle into two grids, one of which has 4×3 rectangles and one of which has 6×5 rectangles; the width and height of these two grids is determined by ℓ .

3.1 An approximation algorithm for problem (*)

In this section we give a factor 2.77 approximation algorithm for problem (*). As a side consequence, it turns out that this algorithm can also be used to give a factor 1.99 approximation algorithm for the continuous k -centers problem in a convex polygon, which we will elaborate on in Section 3.3; this is notable because it is known that there does not exist any approximation algorithm for the k -centers problem in a general metric space with a constant less than 2 unless $P=NP$; see for example [10]. In a nutshell, our algorithm is extremely simple: we first build a bounding box of C that is aligned with its diameter. Next, we attempt to divide this bounding box, which we will call Q , into $n - 1$ rectangles that are as “square” as possible (it will turn out that all of the sub-regions that our algorithm produces will be single points, with the exception of the n -th sub-region, which is the remainder of C). If w and h are the width and height of Q respectively, then a good starting point is to set $p_0 = \lfloor \sqrt{w(n-1)/h} \rfloor$ and $q_0 = \lfloor \sqrt{h(n-1)/w} \rfloor$, and then divide Q into a rectangular $p_0 \times q_0$ grid. Because of the floor functions, it is likely that $p_0 q_0 < n - 1$, which means that we have some “leftover” grid cells at our disposal. We can insert these additional grid cells by either adding them to the columns or the rows of the existing grid. After this step, we let the centers of these $n - 1$ rectangles be the first $n - 1$ sub-regions of our partition of C , and we let R_n be the remaining area of R left over.

Formally, our algorithm is based on a simple scheme which we call subroutine **RectanglePartition**, in which we subdivide an axis-aligned rectangle Q into k smaller rectangles by juxtaposing a pair of rectangular grids. This is described in Algorithm 2 and sketched in Figure 6. Another subroutine that builds off of this, which we call **PointPlacement**, applies Algorithm 2 recursively and determines an “optimal” way to merge two grids together; this is described in Algorithm 3. Finally, our 2.77 approximation algorithm for problem (*) simply applies Algorithm 3 for select values of (p_1, q_1, p_2, q_2) and outputs a partition R_1, \dots, R_n in which the first $n - 1$ sub-regions R_i are each equal to a single point c_i , and the final sub-region R_n is the complement of this, $R_n = C \setminus (\{c_1\} \cup \dots \cup \{c_{n-1}\})$. This is described formally in Algorithm 4 and sketched in Figure 7.

3.2 Analysis of Algorithm 4

This section is devoted to a proof of the following theorem:

Theorem 12. *Algorithm 4 solves problem (*) within a factor of 2.77. Its running time is $\mathcal{O}(m + n \log m)$, where m is the number of edges of the input region.*

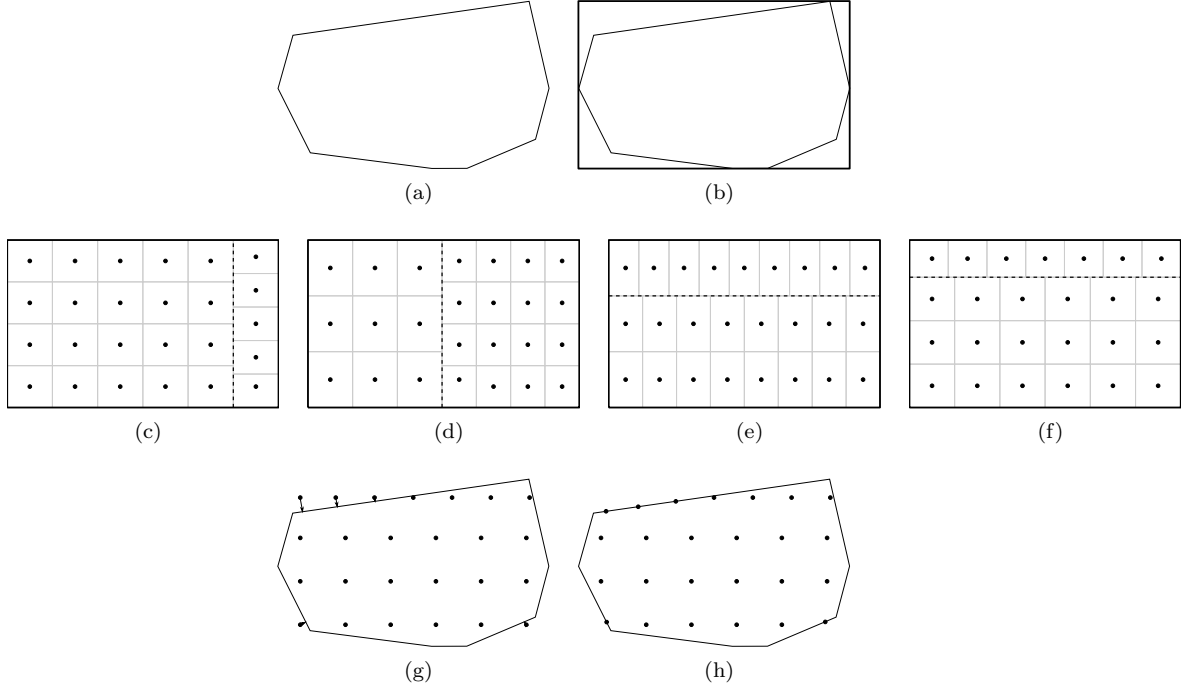


Figure 7: An execution of Algorithm 4 with $n = 26$. We will eventually construct a partition consisting of n sub-regions R_i , in which R_1, \dots, R_{n-1} are each a single point, and R_n their complement. Thus, we will construct $n-1$ rectangles and place a point in each of their centers. In 7a, we orient the input region C to have its diameter be horizontal. We then compute the axis-aligned bounding box Q , as shown in 7b, which happens to have $w/h = 1.62$, which tells us that $p_0 = \lfloor \sqrt{w(n-1)/h} \rfloor = 6$ and $q_0 = \lfloor \sqrt{h(n-1)/w} \rfloor = 3$. Algorithm 4 then constructs various potential point arrangements by positioning two grids alongside each other, four of which are shown in 7c through 7f; for example, in 7d, we break Q into $p = p_0 + 1 = 7$ columns and we have $q = \lfloor (n-1)/p \rfloor = 3$; this gives $s = 4$, so that we break Q into two grids, one of which consists of $(p-s) \times q = 3 \times 3$ rectangles and the other consisting of $s \times (q+1) = 4 \times 4$ rectangles, thus giving $n-1$ rectangles in total. Note that, in each of the four placements, there is always one rectangle \square_j whose associated point c'_j is off-center, as specified in Algorithm 3; this is done to ensure that the connectivity radius of the points placed in Algorithm 4 is at most equal to the maximum of the connectivity radii of each of the two halves (and is only added to make the upcoming proof of the approximation ratio easier). It turns out that the arrangement shown in 7f has the smallest connectivity radius of all the options of Algorithm 4. Thus, we project all points from 7f onto the original polygon C in 7g, and 7h shows the final output of our algorithm: the first $n-1 = 25$ sub-regions are the points that are shown, and the n -th sub-region is their complement.

<p>Input: An axis-aligned rectangle Q, having dimensions $w \times h$, integers p_1, q_1, p_2, q_2, a positive number ℓ, and a “flag” equal to VERTICAL or HORIZONTAL.</p> <p>Output: A partition of Q into $p_1 q_1 + p_2 q_2$ rectangles.</p> <p>if “flag” is VERTICAL then</p> <p style="padding-left: 20px;">Let Q_1 be the left half of Q, having dimensions $(w - \ell) \times h$;</p> <p style="padding-left: 20px;">Let Q_2 be the right half of Q, having dimensions $\ell \times h$;</p> <p style="padding-left: 20px;">Break Q_1 into a $p_1 \times q_1$ rectangular grid, and call the rectangular cells $\square_1, \dots, \square_{p_1 q_1}$;</p> <p style="padding-left: 20px;">Break Q_2 into a $p_2 \times q_2$ rectangular grid, and call the rectangular cells $\square_{p_1 q_1 + 1}, \dots, \square_{p_1 q_1 + p_2 q_2}$;</p> <p>else</p> <p style="padding-left: 20px;">Let Q_1 be the bottom half of Q, having dimensions $w \times (h - \ell)$;</p> <p style="padding-left: 20px;">Let Q_2 be the top half of Q, having dimensions $w \times \ell$;</p> <p style="padding-left: 20px;">Break Q_1 into a $p_1 \times q_1$ rectangular grid, and call the rectangular cells $\square_1, \dots, \square_{p_1 q_1}$;</p> <p style="padding-left: 20px;">Break Q_2 into a $p_2 \times q_2$ rectangular grid, and call the rectangular cells $\square_{p_1 q_1 + 1}, \dots, \square_{p_1 q_1 + p_2 q_2}$;</p> <p>end</p> <p>return $\square_1, \dots, \square_{p_1 q_1 + p_2 q_2}$;</p>
--

Algorithm 2: Algorithm RectanglePartition($Q, p_1, q_1, p_2, q_2, \ell$, “flag”) decomposes rectangle Q into a pair of rectangular grids.

The running time is easy to verify: We require $\mathcal{O}(m)$ time to find the minimum bounding box of C and we require $\mathcal{O}(n)$ time to run Algorithm 2. We can project the $n - 1$ points inside C in $\mathcal{O}(n \log m)$ time by using a point-in-polygon algorithm [14]. We will make some simplifying observations: first, we can disregard the step in Algorithm 4 where the points c'_i are projected onto C because convex projection is always non-expansive [20]; in other words, $\|c'_i - c'_j\| \geq \|c_i - c_j\|$ for all i, j . Secondly, we will assume for simplicity that C has area 1; by construction, this means that the bounding box of Q must have an area between 1 and 2 (this is because the diameter of C is positioned horizontally, i.e. aligned with the long side of box Q). We will assume without loss of generality that Q has area 2 (clearly, larger boxes can only have larger point-to-point distances). Thus, we will say that Q has width w and height $h = 2/w$, with $w \geq h$, i.e. $w \geq \sqrt{2}$. Our approximation ratio will be completely characterized by w and n . Finally, we note that our algorithm always produces a union of two rectangular grids, with a single point that is offset from the others. It is straightforward to verify that the connectivity radius of this union is always at most the connectivity radii of the two halves. Thus, in certain situations below, we will occasionally construct a *single* rectangular grid having the desired approximation ratio (possibly having fewer points than we are permitted), which must therefore be no worse than the union of rectangular grids that Algorithm 4 produces.

We next observe that our algorithm certainly attains the desired approximation ratio whenever $w \geq \sqrt{n}$. This is because one of the configurations produced is to simply divide Q into an $(n - 1) \times 1$ grid of identical rectangles, each having width $w/(n - 1)$ and height $h = 2/w$. Each of the $n - 1$ points c'_i is exactly $w/(n - 1)$ away from its left and right neighbors, and therefore the connectivity radius of R_1, \dots, R_{n-1} is at most $w/(n - 1)$. Furthermore, any point $x_n \in R_n$ must be within a distance of

$$\frac{1}{2} \sqrt{\left(\frac{w}{n-1}\right)^2 + \left(\frac{2}{w}\right)^2}$$

to one of the points c'_i (this is just half the diagonal of any of the $n - 1$ rectangles). Therefore, the approximation ratio is at most the maximum of these two distances, divided by the lower bound of

```

Input: An axis-aligned rectangle  $Q$ , having dimensions  $w \times h$ , integers  $p_1, q_1, p_2, q_2$ , and a “flag” equal to
    VERTICAL or HORIZONTAL.
Output: A collection of  $p_1 q_1 + p_2 q_2$  points in  $Q$ .
Translate  $Q$  so that its bottom left corner is the origin;
if “flag” is VERTICAL then
    Set  $L = \{wp_2/(p_1 + p_2), hp_2/q_1\}$ ;
    Remove any elements  $\ell \in L$  that are not between 0 and  $w$ ;
    /* The first value of  $L$  will give grids whose cells have the same (horizontal) width; the
       second value will give grids in which the (vertical) height of one set of cells is equal
       to the (horizontal) width of the other */
else
    Set  $L = \{hq_2/(q_1 + q_2), wq_2/p_1\}$ ;
    Remove any elements  $\ell \in L$  that are not between 0 and  $h$ ;
    /* The first value of  $L$  will give grids whose cells have the same (vertical) height; the
       second value will give grids in which the (vertical) height of one set of cells is equal
       to the (horizontal) width of the other */
end
Set  $r = \infty$ ;
for  $\ell \in L$  do
    Let  $\square'_1, \dots, \square'_{p_1 q_1 + p_2 q_2} = \text{RectanglePartition}(Q, p_1, q_1, p_2, q_2, \ell, \text{“flag”})$ ;
    Let  $c'_1, \dots, c'_{p_1 q_1 + p_2 q_2}$  be the centers of  $\square'_1, \dots, \square'_{p_1 q_1 + p_2 q_2}$ ;
    Let  $Q_1$  and  $Q_2$  be the two large rectangles that divided  $Q$  in the execution of RectanglePartition;
    if “flag” is VERTICAL then
        Let  $c'_i$  be the center of the bottom right sub-rectangle  $\square_i$  of  $Q_1$ ;
        Let  $c'_j$  be the center of the bottom left sub-rectangle  $\square_j$  of  $Q_2$ ;
        Move  $c'_j$  to be vertically aligned with  $c'_i$ ;
    else
        Let  $c'_i$  be the center of the top left sub-rectangle  $\square_i$  of  $Q_1$ ;
        Let  $c'_j$  be the center of the bottom left sub-rectangle  $\square_j$  of  $Q_2$ ;
        Move  $c'_j$  to be horizontally aligned with  $c'_i$ ;
    end
    Let  $r'$  be the connectivity radius of  $c'_1, \dots, c'_{p_1 q_1 + p_2 q_2}$ ;
    if  $r' < r$  then
        Set  $r = r'$ ;
        Set  $c_1, \dots, c_{p_1 q_1 + p_2 q_2} = c'_1, \dots, c'_{p_1 q_1 + p_2 q_2}$ ;
    end
end
return  $c_1, \dots, c_{p_1 q_1 + p_2 q_2}$ ;

```

Algorithm 3: Algorithm **PointPlacement**($Q, p_1, q_1, p_2, q_2, \text{“flag”}$) places $p_1 q_1 + p_2 q_2$ points inside rectangle Q at the centers of a collection of rectangles.

Input: A convex polygon C and an integer n .
Output: A partition of C into n sub-regions that solves problem (*) within a factor of 2.77.
 Rotate C so that its diameter is aligned with the x -axis;
 Let Q be the axis-aligned bounding box of C ;
 Set $p_0 = \lfloor \sqrt{w(n-1)/h} \rfloor$;
 Set $q_0 = \lfloor \sqrt{h(n-1)/w} \rfloor$;
 Set $r = \infty$;
for $p \in \{p_0 - 1, p_0, p_0 + 1\}$ **do**
 /* break Q into p columns, and then subdivide these columns as evenly as possible */
 Set $q = \lfloor (n-1)/p \rfloor$;
 if $p, q \geq 1$ **then**
 Set $s = (n-1) - pq$, so that $n-1 = pq + s = (p-s)q + s(q+1)$;
 Let $(c'_1, \dots, c'_{n-1}) = \text{PointPlacement}(p-s, q, s, q+1, \text{VERTICAL})$;
 Let r' be the connectivity radius of $c'_1, \dots, c'_{p_1 q_1 + p_2 q_2}$;
 if $r' < r$ **then**
 Set $r = r'$;
 Set $c_1, \dots, c_{p_1 q_1 + p_2 q_2} = c'_1, \dots, c'_{p_1 q_1 + p_2 q_2}$;
 end
 end
end
for $q \in \{q_0 - 1, q_0, q_0 + 1\}$ **do**
 /* break Q into q rows, and then subdivide these rows as evenly as possible */
 Set $p = \lfloor (n-1)/q \rfloor$;
 if $p, q \geq 1$ **then**
 Set $s = (n-1) - pq$, so that $n-1 = pq + s = p(q-s) + (p+1)s$;
 Let $(c_1, \dots, c_{n-1}) = \text{PointPlacement}(p, q-s, p+1, s, \text{HORIZONTAL})$;
 Let r' be the connectivity radius of $c'_1, \dots, c'_{p_1 q_1 + p_2 q_2}$;
 if $r' < r$ **then**
 Set $r = r'$;
 Set $c_1, \dots, c_{p_1 q_1 + p_2 q_2} = c'_1, \dots, c'_{p_1 q_1 + p_2 q_2}$;
 end
 end
end
 Project each point c_i onto the polygon C (if it is not already inside C);
 Let $R_i = \{c_i\}$ for $i \in \{1, \dots, n-1\}$ and let $R_n = C \setminus (\{c_1\} \cup \dots \cup \{c_{n-1}\})$;
return R_1, \dots, R_n ;

Algorithm 4: Algorithm RegionPartition(C, n) partitions convex polygon C into n sub-regions.

w/n (from Theorem 11) :

$$\begin{aligned}
\text{Rat} &\leq \frac{\max \left\{ \frac{w}{n-1}, \frac{1}{2} \sqrt{\left(\frac{w}{n-1} \right)^2 + \left(\frac{2}{w} \right)^2} \right\}}{w/n} = \max \left\{ \frac{n}{n-1}, \frac{\frac{1}{2} \sqrt{\left(\frac{w}{n-1} \right)^2 + \left(\frac{2}{w} \right)^2}}{w/n} \right\} \\
&\leq \max \left\{ 2, \frac{\frac{1}{2} \left[\left(\frac{w}{n-1} \right) + \left(\frac{2}{w} \right) \right]}{w/n} \right\} \\
&\leq \max \left\{ 2, \frac{\frac{1}{2} \left[\left(\frac{w}{n-1} \right) + \left(\frac{2}{\sqrt{n}} \right) \right]}{w/n} \right\} \\
&\leq \max \left\{ 2, \frac{n}{2(n-1)} + 1 \right\} \leq 2 < 2.77 \text{ for all } n \geq 2
\end{aligned}$$

as desired. We are therefore permitted to assume that $w < \sqrt{n}$ throughout.

We will next verify computationally that the approximation ratio of 2.77 holds for $n \leq 20$. Note that for any fixed n , we can consider only those values of w in the finite interval $[\sqrt{2}, \sqrt{n})$. These ratios are shown in the plot in Figure 8. Since all values on that plot are less than 2.77, we obtain the desired result, and we are therefore free to assume that $w < \sqrt{n}$ and that $n \geq 21$. Note that we can now improve the result from the previous paragraph to conclude that our approximation ratio holds whenever $w \geq 0.61\sqrt{n}$. This is because we can safely assume that $n \geq 21$, and because the connectivity radius associated with an $(n-1) \times 1$ grid of identical rectangles satisfies

$$\begin{aligned}
\text{Rat} &\leq \frac{\max \left\{ \frac{w}{n-1}, \frac{1}{2} \sqrt{\left(\frac{w}{n-1} \right)^2 + \left(\frac{2}{w} \right)^2} \right\}}{w/n} = \max \left\{ \frac{n}{n-1}, \frac{\frac{1}{2} \sqrt{\left(\frac{w}{n-1} \right)^2 + \left(\frac{2}{w} \right)^2}}{w/n} \right\} \\
&\leq \max \left\{ \frac{21}{20}, \frac{\frac{1}{2} \sqrt{\left(\frac{21}{20} \right)^2 \left(\frac{w}{n} \right)^2 + \left(\frac{2}{w} \right)^2}}{w/n} \right\} \\
&= \max \left\{ \frac{21}{20}, \frac{\sqrt{441w^4 + 1600n^2}}{40w^2} \right\} \\
&\leq \max \left\{ \frac{21}{20}, \frac{\sqrt{441w^4 + 1600(w/0.61)^4}}{40w^2} \right\} < 2.77
\end{aligned}$$

as desired. Thus, we will now assume that $n \geq 21$ and $w < 0.61\sqrt{n}$.

The assumption that $w < 0.61\sqrt{n}$ and that $n \geq 21$ implies that the number of rows, q_0 , that are used in Algorithm 4 is

$$q_0 = \left\lfloor \sqrt{\frac{h(n-1)}{w}} \right\rfloor = \left\lfloor \sqrt{\frac{2(n-1)}{w^2}} \right\rfloor = \left\lfloor \sqrt{\frac{2n}{w^2} \cdot \frac{n-1}{n}} \right\rfloor \geq \left\lfloor \sqrt{\frac{2n}{w^2} \cdot \frac{20}{21}} \right\rfloor > \lfloor 2.27 \rfloor = 2;$$

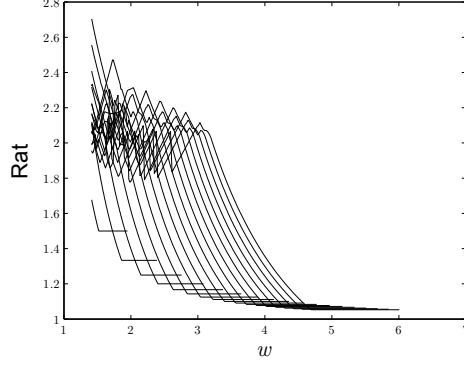


Figure 8: The approximation ratios realized by Algorithm 4, for $n \in \{2, \dots, 20\}$ and $w \in [\sqrt{2}, \frac{4}{3}\sqrt{n}]$. These are obtained by dividing the connectivity radius for the output solution by the maximum of the two lower bounds (1) and (2). Note that it would have sufficed to compute these ratios for the smaller interval $w \in [\sqrt{2}, \sqrt{n}]$, by our previous analysis; we merely compute the ratio for this slightly longer interval for purposes of clarity.

We can divide Q into $\lfloor (n-1)/q_0 \rfloor \times q_0$ sub-rectangles and place a point c'_i in the center of each of these. Note that by definition of the floor function, we always have

$$q_0 \leq \sqrt{\frac{h(n-1)}{w}} = \sqrt{\frac{2(n-1)}{w^2}} < q_0 + 1,$$

or equivalently

$$\frac{\sqrt{2(n-1)}}{q_0 + 1} < w \leq \frac{\sqrt{2(n-1)}}{q_0}. \quad (3)$$

The distance between any vertical neighbors is obviously $h/q_0 = 2/(q_0 w)$ and the distance between any horizontal neighbors is at most

$$\frac{w}{\lfloor (n-1)/q_0 \rfloor} \leq \frac{w}{(n-1)/q_0 - 1} = \frac{q_0 w}{(n-1) - q_0}$$

and therefore, applying lower bound (1), the approximation ratio is at most

$$\text{Rat} \leq \frac{\max \left\{ \frac{2}{q_0 w}, \frac{q_0 w}{(n-1) - q_0} \right\}}{\sqrt{\frac{1}{\pi + (n-1)(\pi/3 + \sqrt{3}/2)}}}.$$

Assume that $q_0 \geq 3$; we address the case where $q_0 = 2$ in Section A of the online supplement. The first term of the $\max\{\cdot, \cdot\}$ expression is largest when w is as small as possible, which occurs when

$w = \sqrt{2(n-1)}/(q_0+1)$; the above ratio is then equal to

$$\begin{aligned}
\frac{\frac{2}{q_0 w}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} &= \frac{\frac{2}{q_0 \sqrt{2(n-1)/(q_0+1)}}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} \\
&= \frac{q_0+1}{q_0} \cdot \frac{\sqrt{(9\sqrt{3}+6\pi)(n-1)+18\pi}}{3\sqrt{n-1}} \\
&\leq \frac{4}{3} \cdot \frac{\sqrt{(9\sqrt{3}+6\pi)(n-1)+18\pi}}{3\sqrt{n-1}} < 2.77 \text{ for } n \geq 21.
\end{aligned}$$

The second term of the $\max\{\cdot, \cdot\}$ expression is largest when w is as large as possible, which occurs when $w = \sqrt{2(n-1)}/q_0$; the above ratio is then equal to

$$\begin{aligned}
\frac{\frac{q_0 w}{(n-1)-q_0}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} &= \frac{\frac{\sqrt{2(n-1)}}{(n-1)-q_0}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} \\
&\leq \frac{\frac{\sqrt{2(n-1)}}{(n-1)-\sqrt{n-1}}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} \\
&= \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{(3\sqrt{3}+2\pi)(n-1)+6\pi}}{\sqrt{n-1}-1} < 2.77 \text{ for all } n \geq 21,
\end{aligned}$$

where we have used the fact that $q_0 \leq \sqrt{n-1}$, which holds because we have $h \leq w$ and we define $q_0 = \lfloor \sqrt{h(n-1)}/w \rfloor$. This completes the proof for the case where $q_0 \neq 2$ and we refer the reader to Section A of the online supplement for the remaining analysis.

3.3 An approximation algorithm for the continuous k -centers problem in a convex polygon

In the k -centers problem, we are given a domain \mathcal{D} equipped with a distance function $\delta(\cdot, \cdot)$ and our objective is to place k points x_1, \dots, x_k in \mathcal{D} to minimize the maximum distance between any point $x \in \mathcal{D}$ and its nearest neighbor x_i ; that is, the problem is

$$\text{minimize}_{\{x_1, \dots, x_k\} \subset \mathcal{D}} \max_{x \in \mathcal{D}} \min_i \delta(x, x_i). \quad (**)$$

In this section we consider the special case where \mathcal{D} is a convex polygon C in the plane and $\delta(\cdot, \cdot)$ is the Euclidean norm, and we show how the rectangular partitioning scheme of Algorithm 2 can be applied to design an approximation algorithm with factor 1.99. This is notable because it is known that there does not exist any approximation algorithm for the problem (**) in a general metric space with a constant less than 2 unless $P=NP$; see for example [10]. We require two extremely simple lower bounds:

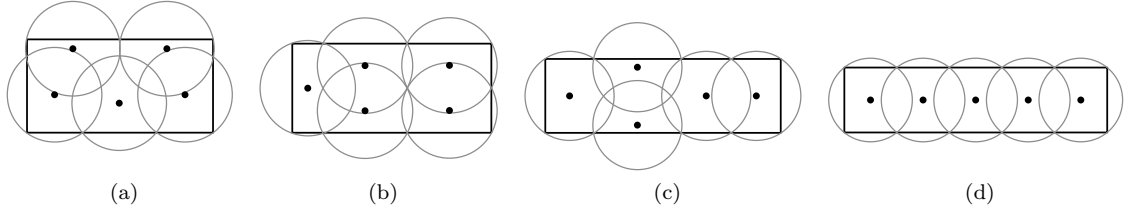


Figure 9: The optimal solutions to the 5-center problem in a rectangle, which depend on the dimensions of said rectangle as shown above; the configurations are described formally in [11]. These are necessary for the “very particular edge case” described in the end of Algorithm 5.

Lemma 13. *The optimal objective value r^* to problem (**), where \mathcal{D} is a convex polygon C with area A and diameter d and the distance function $\delta(\cdot, \cdot)$ is the Euclidean norm, satisfies*

$$r^* \geq \max \left\{ \sqrt{\frac{A}{\pi k}}, \frac{d}{2k} \right\}.$$

Proof. This is straightforward: if x_1^*, \dots, x_k^* are an optimal solution, then if we center a ball of radius r^* at each x_i^* , we must cover all of C , therefore $A \leq k\pi(r^*)^2$. The second bound arises from the observation that the longest line segment in C (whose length is d) must be covered by the k points, and each point is capable of covering a length of at most $2r^*$, whence $d \leq 2kr^*$. \square

The intuition behind the approximation factor of 1.99 is as follows: assume without loss of generality that $A = 1$, and recall from before that we can enclose C in a box Q whose area is at most 2. Suppose that we somehow divide Q into k rectangles, each having area equal to $2/k$, and suppose that the aspect ratios of these rectangles (i.e. the ratios of the long side to the short side) do not exceed 2. If we place points x_1, \dots, x_k at the centers of these rectangles, then the distance between any point $x \in C \subseteq Q$ and its nearest center x_i is at most half of the diagonal of these rectangles, which (by our bounded aspect ratio) is at most $\frac{1}{2}\sqrt{(1/\sqrt{k})^2 + (2/\sqrt{k})^2} = \frac{1}{2}\sqrt{5/k}$. On the other hand, our first lower bound says that $r^* \geq \sqrt{1/(\pi k)}$ and thus the approximation ratio is at most

$$\text{Rat} \leq \frac{\frac{1}{2}\sqrt{5/k}}{\sqrt{1/(\pi k)}} \approx 1.982 < 1.99.$$

Our algorithm is described in Algorithm 5 and sketched in Figure 10.

3.4 Analysis of Algorithm 5

This section is similar to Section 3.2 and is devoted to a proof of the following theorem:

Theorem 14. *Algorithm 5 solves the continuous k -centers problem in a convex polygon C within a factor of 1.99 for any $k \geq 6$. Its running time is $\mathcal{O}(m + n \log m)$, where m is the number of edges of the input region.*

The claimed running time is true for the same reasons as in the proof of Theorem 12. We make the same simplifying observations as in Section 3.2, namely that:

Input: A convex polygon C and an integer $k \geq 6$.
Output: The locations of k points inside C to solve the continuous k -centers problem in C within a factor of 1.99.

Rotate C so that its diameter is aligned with the x -axis;
Let Q be the axis-aligned bounding box of C ;
Set $p_0 = \lfloor \sqrt{wk/h} \rfloor$;
Set $q_0 = \lfloor \sqrt{hk/w} \rfloor$;
Set $r = \infty$;
for $p \in \{p_0 - 1, p_0, p_0 + 1\}$ **do**
 /* break Q into p columns, and then subdivide these columns as evenly as possible */
 Set $q = \lfloor k/p \rfloor$;
 if $p, q \geq 1$ **then**
 Set $s = k - pq$, so that $k = pq + s = (p - s)q + s(q + 1)$;
 Let ℓ be the solution to $\left(\frac{w-\ell}{p-s}\right)^2 + \left(\frac{h}{q}\right)^2 = \left(\frac{\ell}{s}\right)^2 + \left(\frac{h}{q+1}\right)^2$ that satisfies $0 \leq \ell \leq w$; if no such ℓ exists, let $\ell = w$;
 Let $\square_1, \dots, \square_k = \text{RectanglePartition}(Q, p - s, q, s, q + 1, \ell, \text{VERTICAL})$;
 /* By construction of ℓ , all boxes have the same diagonal length if $\ell < w$ */
 Let (x'_1, \dots, x'_k) be the centers of the boxes \square_i ;
 Let $r' = \max_{x \in C} \min_i \|x - x'_i\|$;
 if $r' < r$ **then**
 Set $r = r'$;
 Set $x_1, \dots, x_k = x'_1, \dots, x'_k$;
 end
 end
end
for $q \in \{q_0 - 1, q_0, q_0 + 1\}$ **do**
 /* break Q into q rows, and then subdivide these rows as evenly as possible */
 Set $p = \lfloor k/q \rfloor$;
 if $p, q \geq 1$ **then**
 Set $s = k - pq$, so that $k = pq + s = p(q - s) + (p + 1)s$;
 Let ℓ be the solution to $\left(\frac{w}{p}\right)^2 + \left(\frac{h-\ell}{q-s}\right)^2 = \left(\frac{w}{p+1}\right)^2 + \left(\frac{\ell}{s}\right)^2$ that satisfies $0 \leq \ell \leq h$; if no such ℓ exists, let $\ell = h$;
 Let $\square_1, \dots, \square_k = \text{RectanglePartition}(Q, p, q - s, p + 1, s, \ell, \text{HORIZONTAL})$;
 /* By construction of ℓ , all boxes have the same diagonal length if $\ell < h$ */
 Let (x'_1, \dots, x'_k) be the centers of the boxes \square_i ;
 Let $r' = \max_{x \in C} \min_i \|x - x'_i\|$;
 if $r' < r$ **then**
 Set $r = r'$;
 Set $x_1, \dots, x_k = x'_1, \dots, x'_k$;
 end
 end
end
if $q_0 = 1$ and k is odd **then**
 /* This is a very particular edge case that requires special attention */
 Set $\ell = 11.08/\sqrt{k} - 6.10/w$ and divide Q into two rectangles Q_1 and Q_2 with dimensions $\ell \times h$ and $(w - \ell) \times h$ respectively;
 Place (x'_1, \dots, x'_k) as follows: put 5 points in Q_1 according to Figure 9 and place $k - 5$ points in a $(k - 5)/2 \times 2$ grid in Q_2 ;
 Let $r' = \max_{x \in C} \min_i \|x - x'_i\|$;
 if $r' < r$ **then**
 Set $r = r'$;
 Set $x_1, \dots, x_k = x'_1, \dots, x'_k$;
 end
end
Project each point x_i onto the polygon C (if it is not already inside C);
return x_1, \dots, x_k ;

Algorithm 5: Algorithm $\text{KCenters}(C, k)$ places k points inside C .

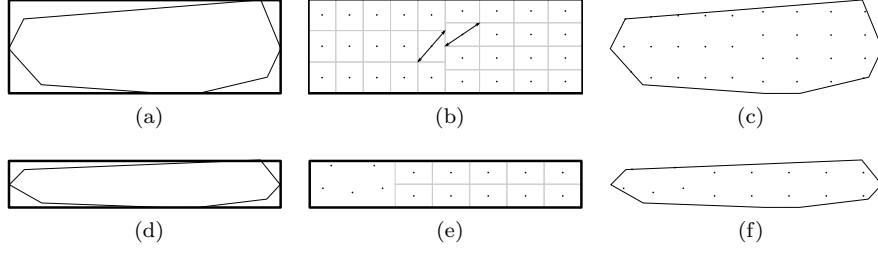


Figure 10: Two executions of Algorithm 5 with $k = 31$ and $k = 15$. We start with a convex polygon with a horizontal diameter and a bounding box Q in 10a. We then apply the rectangular partitioning Algorithm 2 for various input values, and the best set of inputs is shown in 10b; the two arrows indicate that all rectangles have the same diagonal lengths. Figure 10c then shows the final output. Figures 10d through 10f show the same thing for a different polygon and $k = 15$, except that we also encounter the “very particular edge case” described in the end of Algorithm 5 in which we place 5 points according to Figure 9.

- We disregard the projection of the points x'_i onto C .
- We assume that C has area 1.
- We assume that C is oriented so that its diameter is horizontal and that its bounding box Q has area 2, and hence Q has width w and height $h = 2/w$, with $w \geq h$, i.e. $w \geq \sqrt{2}$.

We next observe that our algorithm certainly attains the desired approximation ratio whenever $w \geq \frac{3}{2}\sqrt{k}$. This is because one of the configurations produced is to simply divide Q into a $k \times 1$ grid of identical rectangles, each having width w/k and height $h = 2/w$, and place each x'_i in the middle of these. Any point $x \in C \subseteq Q$ must be within a distance of

$$\frac{1}{2} \sqrt{\left(\frac{w}{k}\right)^2 + \left(\frac{2}{w}\right)^2}$$

to one of the points x'_i and therefore (using the lower bound $r^* \geq w/(2k)$) the approximation ratio is at most

$$\begin{aligned} \text{Rat} &\leq \frac{\frac{1}{2} \sqrt{\left(\frac{w}{k}\right)^2 + \left(\frac{2}{w}\right)^2}}{w/(2k)} \\ &\leq \frac{\frac{1}{2}(w/k + 2/w)}{w/(2k)} = 1 + 2k/w^2 < 1.99 \end{aligned}$$

as desired. By using the lower bound $r^* \geq 1/\sqrt{\pi k}$, our algorithm also attains the 1.99 approximation ratio whenever $\sqrt{k} \leq w \leq 2\sqrt{k}$ because the same $k \times 1$ configuration has an approximation ratio of

$$\begin{aligned} \text{Rat} &\leq \frac{\frac{1}{2} \sqrt{\left(\frac{w}{k}\right)^2 + \left(\frac{2}{w}\right)^2}}{1/\sqrt{\pi k}} \\ &= \frac{\sqrt{\pi}}{2} \cdot \sqrt{t^2 + 4/t^2} < 1.99 \end{aligned}$$

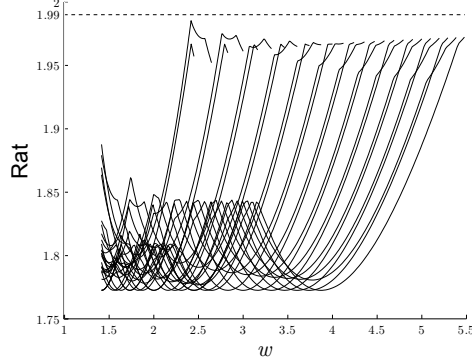


Figure 11: The approximation ratios realized by Algorithm 5, for $k \in \{6, \dots, 30\}$ and $w \in [\sqrt{2}, \sqrt{k}]$. These are obtained by dividing the objective value of the output solution (i.e. half of the largest diagonal of all the rectangles) by the maximum of the two lower bounds from Theorem 13.

where we set $t = w/\sqrt{k}$. Thus, from now on, we will assume that $w < \sqrt{k}$.

We will next verify computationally that the approximation ratio of 1.99 holds for $k \leq 30$. Note that for any fixed k , we can consider only those values of w in the finite interval $[\sqrt{2}, \sqrt{k}]$. These ratios are shown in the plot in Figure 11. Since all values on that plot are less than 1.99, we obtain the desired result, and we are therefore free to assume that $w < \sqrt{k}$ and that $k \geq 31$.

The remainder of our analysis is a case-by-case study where we consider the different values of $q_0 = \lfloor \sqrt{hk}/w \rfloor = \lfloor \sqrt{2k}/w \rfloor$. Because $w \geq h$, it is of course always true that $q_0 \leq p_0$. Suppose that $q_0 \geq 8$, whence $p_0 \geq 8$; we can partition Q into $p_0 q_0$ rectangles, and each of these rectangles has an aspect ratio of at most $(q_0 + 1)/q_0 \leq 9/8$. Thus, each rectangle has area $2/(p_0 q_0)$ and aspect ratio at most $9/8$, and therefore each rectangle's diagonal is at most $\frac{1}{6} \sqrt{\frac{145}{p_0 q_0}}$. The distance from any point $x \in C \subseteq Q$ to the center of the rectangle containing it is at most half of this. By construction, we know that $k = p_0 q_0 + s$, where $s \leq p_0 + q_0$. Thus, our approximation ratio is at most

$$\begin{aligned} \text{Rat} &\leq \frac{\frac{1}{12} \sqrt{\frac{145}{p_0 q_0}}}{1/\sqrt{\pi k}} = \frac{\frac{1}{12} \sqrt{\frac{145}{p_0 q_0}}}{1/\sqrt{\pi(p_0 q_0 + s)}} \\ &\leq \frac{\frac{1}{12} \sqrt{\frac{145}{p_0 q_0}}}{1/\sqrt{\pi(p_0 q_0 + p_0 + q_0)}} < 1.779 \sqrt{1 + \frac{1}{p_0} + \frac{1}{q_0}} < 1.99 \end{aligned}$$

since $p_0, q_0 \geq 8$. Thus, we can assume from now on that $q_0 \in \{0, \dots, 7\}$. In fact, since we have already concluded that our ratio holds when $w \geq \sqrt{k}$, we see that the case where $q_0 = 0$ is already taken care of (this is because $q_0 = 0 \iff \sqrt{2k}/w < 1 \iff w > \sqrt{2k}$). Therefore, we will now consider the case where $q_0 \in \{1, \dots, 7\}$ and $n \geq 31$, which will complete the proof. We will assume that $q_0 \geq 2$ and address the case where $q_0 = 1$ in Section B of the online supplement.

If $q_0 \geq 2$, we will show that the desired approximation ratio holds when we divide Q into a grid consisting of $\lfloor k/q_0 \rfloor \times q_0$ rectangles or a grid of $\lfloor k/(q_0 + 1) \rfloor \times (q_0 + 1)$ rectangles. If we use a $\lfloor k/q_0 \rfloor \times q_0$ grid, then each sub-rectangle has dimensions

$$\frac{w}{\lfloor k/q_0 \rfloor} \times \frac{h}{q_0} = \frac{w}{\lfloor k/q_0 \rfloor} \times \frac{2}{q_0 w}$$

and half of the diagonal of each sub-rectangle (which, as we have already seen, is the largest distance between a point $x \in C \subseteq Q$ and its nearest neighbor x_i) is

$$\begin{aligned} \frac{1}{2} \sqrt{\left(\frac{w}{\lfloor k/q_0 \rfloor}\right)^2 + \left(\frac{2}{q_0 w}\right)^2} &\leq \frac{1}{2} \sqrt{\left(\frac{w}{k/q_0 - 1}\right)^2 + \left(\frac{2}{q_0 w}\right)^2} \\ &= \sqrt{\frac{q_0^2 w^2}{4(k - q_0)^2} + \frac{1}{q_0^2 w^2}} = \sqrt{\frac{t^2}{2kq_0^2} + \frac{kq_0^2}{2(k - q_0)^2 t^2}} \end{aligned} \quad (4)$$

where we define $t = \sqrt{2k}/w$, which must satisfy $q_0 \leq t < q_0 + 1$. We similarly find that the $\lfloor k/(q_0 + 1) \rfloor \times (q_0 + 1)$ grid gives rectangles whose half-diagonals are at most

$$\frac{1}{2} \sqrt{\left(\frac{w}{\lfloor k/(q_0 + 1) \rfloor}\right)^2 + \left(\frac{2}{(q_0 + 1)w}\right)^2} \leq \sqrt{\frac{t^2}{2k(q_0 + 1)^2} + \frac{k(q_0 + 1)^2}{2(k - q_0 - 1)^2 t^2}}. \quad (5)$$

We are therefore interested in showing that the minimum of (4) and (5) is bounded above by $1.99/\sqrt{\pi k}$ for $q_0 \in \{2, \dots, 7\}$ and all $k \geq 31$. Suppose that q_0 and k are fixed, and consider (4) and (5) as functions of t ; since each of these is the square root of a convex function in t , they are both maximized when t is as large or as small as possible, i.e. $t = q_0$ or $t = q_0 + 1$. The function of t defined by taking the minimum of (4) and (5) must be maximized either at one of these values, or at a value of t such that (4) and (5) are equal. Such values of t can be computed analytically because both terms are of the form $\sqrt{\alpha t^2 + \beta/t^2}$ for constant terms α, β . Thus, we can restrict ourselves to a finite set of values $t \in T$, and our goal is to show that

$$\frac{\min \left\{ \sqrt{\frac{t^2}{2kq_0^2} + \frac{kq_0^2}{2(k - q_0)^2 t^2}}, \sqrt{\frac{t^2}{2k(q_0 + 1)^2} + \frac{k(q_0 + 1)^2}{2(k - q_0 - 1)^2 t^2}} \right\}}{1/\sqrt{\pi k}} \leq 1.99$$

for all $k \geq 31$, $q_0 \in \{2, \dots, 7\}$, and all $t \in [q_0, q_0 + 1)$. By the preceding argument, we can remove the dependency on t , so that we must show that

$$\max_{t \in T} \frac{\min \left\{ \sqrt{\frac{t^2}{2kq_0^2} + \frac{kq_0^2}{2(k - q_0)^2 t^2}}, \sqrt{\frac{t^2}{2k(q_0 + 1)^2} + \frac{k(q_0 + 1)^2}{2(k - q_0 - 1)^2 t^2}} \right\}}{1/\sqrt{\pi k}} \leq 1.99 \quad (6)$$

which now depends only on q_0 and k (since the finite set T is determined by q_0 and k). Since there are only 6 different values of q_0 that are of interest, we therefore are left with 6 different univariate functions of k alone, and these are shown in Figure 12. It is entirely straightforward (albeit tedious) to verify algebraically that the desired results hold, and we omit this for brevity. This completes the proof for the case where $q_0 \neq 1$ and we refer the reader to Section B of the online supplement for the remaining analysis.

4 Problem (*) with a convexity constraint

Although Algorithm 4 does not have the same “bottleneck” problem as Algorithm 1 (as described in the end of Section 2), it of course has deficiencies of its own: first of all, the sub-regions R_1, \dots, R_n have irregular shapes, and secondly, the areas of these sub-regions are extremely unbalanced (all of

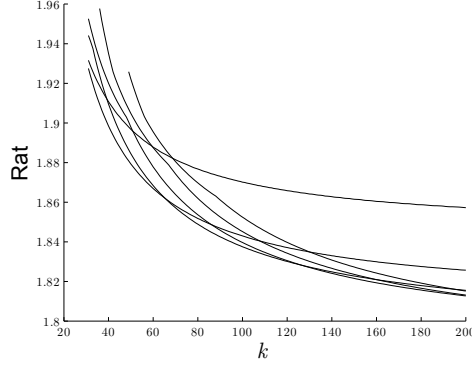


Figure 12: The value of the left-hand side of (6), for $q_0 \in \{2, \dots, 7\}$ and $k \geq \max\{q_0^2, 31\}$. It is entirely straightforward (albeit tedious) to verify algebraically that the ratio is always less than 1.99.

the area is allocated to the single region R_n except for a finite set of $n-1$ points). Thus, we propose Algorithm 6, which remedies both of these issues by producing sub-regions that are all convex and that all have areas that do not exceed $\frac{22}{9}A/n = 2.4A/n$ (recall that Algorithm 1 produced convex sub-regions whose areas were equal to A/n). As in Section 3, Algorithm 6 is based on a sequence of calls to the `RectanglePartition` subroutine, Algorithm 2, and is sketched in Figure 13. This section is devoted to a proof of the following theorem:

Theorem 15. *Algorithm 6 is a 5.94 approximation algorithm for problem (*) with an additional constraint that all sub-regions R_i be convex. In addition, all sub-regions output by Algorithm 6 have area of at most $\frac{22}{9}A/n = 2.4A/n$. Its running time is $\mathcal{O}(m + n \log m)$, where m is the number of edges of the input region.*

The claimed running time is true for the same reasons as in the proof of Theorem 12. This proof turns out to be much shorter than the preceding results. We will use the lower bounds from Theorems 9 and 11. The following result bounds the output of Algorithm 6 from above:

Lemma 16. *Let C be a convex polygon and let R_1, \dots, R_n be the output of Algorithm 6. If R_1, \dots, R_n is obtained by intersecting a single $p \times q$ grid with C , then the connectivity radius r satisfies*

$$r \leq \max \left\{ \sqrt{(2w/p)^2 + (h/q)^2}, \sqrt{(w/p)^2 + (2h/q)^2} \right\},$$

and if R_1, \dots, R_n is obtained by merging two grids together, with sub-rectangles having dimensions $w_i \times h_i$ for $i \in \{1, 2\}$, then r satisfies

$$r \leq \max \left\{ \sqrt{(2w_i)^2 + h_i^2}, \sqrt{w_i^2 + (2h_i)^2}, \sqrt{(w_1 + w_2)^2 + (h_1 + h_2)^2} \right\} \text{ for } i \in \{1, 2\}.$$

Proof. This follows from basic facts about convexity; the third term in the second inequality, $\sqrt{(w_1 + w_2)^2 + (h_1 + h_2)^2}$, guarantees that the two grids are connected to each other. \square

Our proof of Theorem 15 now follows: as in all of the previous analyses, we assume that C has area 1 and that Q has area 2. We first observe that our approximation ratio certainly holds

Input: A convex polygon C and an integer n .
Output: A partition of C into at most n convex sub-regions that solves problem (*) within a factor of 5.94.
 Rotate C so that its diameter is aligned with the x -axis;
 Let Q be the axis-aligned bounding box of C ;
 Set $p_0 = \lfloor \sqrt{wn/h} \rfloor$;
 Set $q_0 = \lfloor \sqrt{hn/w} \rfloor$;
 Set $r = \infty$;
for $p \in \{p_0, p_0 + 1\}$ **do**
 /* break Q into p columns, and then subdivide these columns as evenly as possible */
 Set $q = \lfloor n/p \rfloor$;
 if $p, q \geq 1$ **then**
 /* Try a simple subdivision into identical rectangles first */
 Let $\square_1, \dots, \square_{pq}$ be a subdivision of Q into a $p \times q$ grid (with each cell having dimensions $w/p \times h/q$);
 Let r' be the connectivity radius of $\square_1, \dots, \square_{pq}$;
 if $r' < r$ **then**
 Set $r = r'$;
 Set $R_i = \square_i \cap C$ for each $i \in \{1, \dots, pq\}$;
 end
 /* Try using the RectanglePartition subroutine */
 Set $s = n - pq$, so that $n = pq + s = (p - s)q + s(q + 1)$;
 Set $\ell = \frac{ws(q+1)}{pq+s}$;
 Let $\square_1, \dots, \square_n = \text{RectanglePartition}(Q, p - s, q, s, q + 1, \ell, \text{VERTICAL})$;
 /* By construction of ℓ , all boxes \square_i have equal area */
 Let r' be the connectivity radius of $\square_1, \dots, \square_n$;
 if $r' < r$ **then**
 Set $r = r'$;
 Set $R_i = \square_i \cap C$ for each $i \in \{1, \dots, n\}$;
 end
 end
end
for $q \in \{q_0, q_0 + 1\}$ **do**
 /* break Q into q rows, and then subdivide these rows as evenly as possible */
 Set $p = \lfloor n/q \rfloor$;
 if $p, q \geq 1$ **then**
 /* Try a simple subdivision into identical rectangles first */
 Let $\square_1, \dots, \square_{pq}$ be a subdivision of Q into a $p \times q$ grid (with each cell having dimensions $w/p \times h/q$);
 Let r' be the connectivity radius of $\square_1, \dots, \square_{pq}$;
 if $r' < r$ **then**
 Set $r = r'$;
 Set $R_i = \square_i \cap C$ for each $i \in \{1, \dots, pq\}$;
 end
 /* Try using the RectanglePartition subroutine */
 Set $s = n - pq$, so that $n = pq + s = p(q - s) + (p + 1)s$;
 Set $\ell = \frac{hs(p+1)}{pq+s}$;
 Let $\square_1, \dots, \square_n = \text{RectanglePartition}(Q, p, q - s, p + 1, s, \ell, \text{HORIZONTAL})$;
 Let r' be the connectivity radius of $\square_1, \dots, \square_n$;
 if $r' < r$ **then**
 Set $r = r'$;
 Set $R_i = \square_i \cap C$ for each $i \in \{1, \dots, n\}$;
 end
 end
end
return R_1, \dots, R_n ;

Algorithm 6: Algorithm ConvexPartition(C, n) partitions convex polygon C into at most n convex sub-regions.

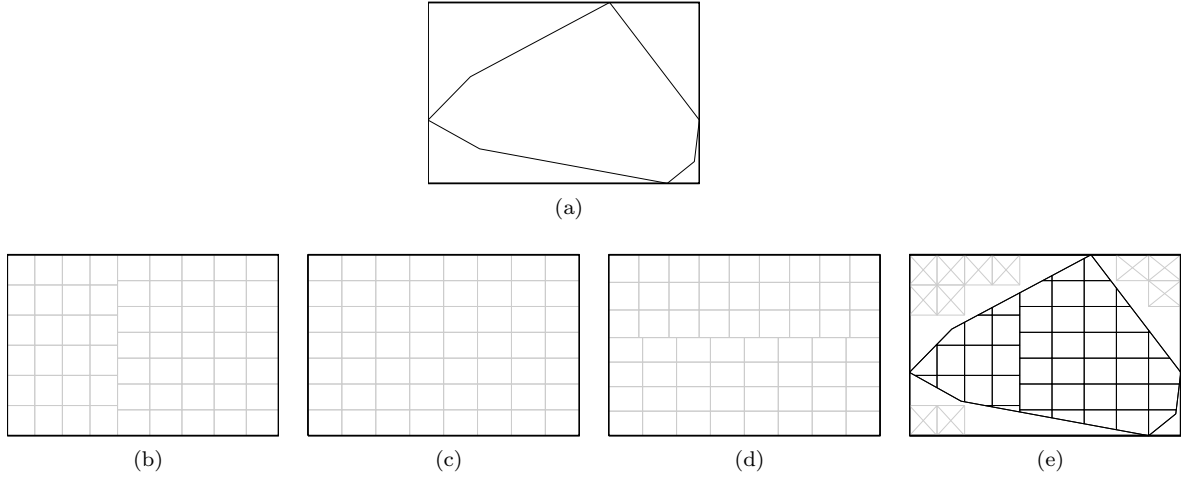


Figure 13: An execution of Algorithm 6 with $n = 59$. We start with a convex polygon, which has been oriented in 13a to have its diameter be horizontal, embedded in bounding box Q . Algorithm 6 then constructs various potential convex partitions consisting of either a single grid or a pair of grids; three such partitions are shown in 13b through 13d. The partition in 13b has the smallest connectivity radius, and thus the output partition is shown in 13e. Note that there are 11 empty sub-regions, as indicated by the crossed-out boxes. Thus, the final output partition consists of 48 sub-regions.

whenever $w \geq \frac{3}{5}\sqrt{n}$, because the connectivity radius of an $n \times 1$ grid is at most $\sqrt{(2w/n)^2 + h^2}$ and therefore the approximation ratio is

$$\begin{aligned} \text{Rat} &\leq \frac{\sqrt{(2w/n)^2 + h^2}}{w/n} = \frac{\sqrt{(2w/n)^2 + (2/w)^2}}{w/n} \\ &= \frac{2\sqrt{w^4 + n^2}}{w^2} \leq \frac{2\sqrt{w^4 + \frac{625}{81}w^4}}{w^2} < 5.94 \end{aligned}$$

as desired (and certainly, each sub-region R_i has area equal to $2/n < \frac{22}{9n}$). We will next verify computationally that the approximation ratio of 5.94 holds and that all sub-regions have area of at most $\frac{22}{9n}$ for $n \leq 32$ (we choose a large threshold value of n solely to make this proof as short as possible). Note that for any fixed n , we can consider only those values of w in the finite interval $[\sqrt{2}, \frac{3}{5}\sqrt{n}]$. These ratios are shown in the plot in Figure 14. Since all values on that plot are less than 5.94, we obtain the desired result, and we are therefore free to assume that $w < \frac{3}{5}\sqrt{n}$ and that $n \geq 33$.

We next consider the case where $q_0 = \lfloor \sqrt{hn/w} \rfloor \geq 4$, or equivalently, $w \leq \sqrt{2n}/4$. Suppose that we partition Q into a $\lfloor n/q_0 \rfloor \times q_0$ grid; the connectivity radius of such a grid is certainly bounded above by that of a $p_0 \times q_0$ grid because $p_0 \geq \lfloor n/q_0 \rfloor$ by construction. If we use a $p_0 \times q_0$ grid, then we know that each of the (identical) grid cells has an aspect ratio of at most $5/4$ because $p_0, q_0 \geq 4$. The union of any two adjacent rectangles therefore has an aspect ratio of at most $5/2$. Thus, the connectivity radius of such a configuration is at most equal to the diagonal length of this union of

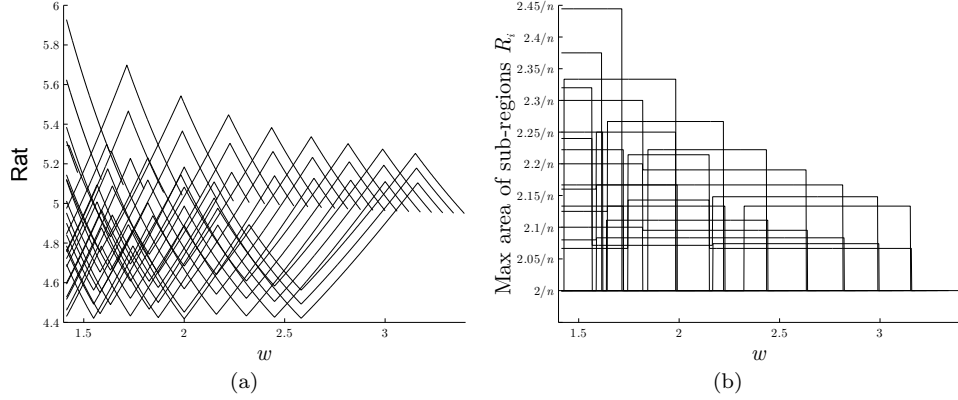


Figure 14: The approximation ratios realized by Algorithm 6, for $n \in \{2, \dots, 32\}$ and $w \in [\sqrt{2}, \frac{3}{5}\sqrt{n}]$ and the maximum areas of the sub-regions that are output.

rectangles, which by routine calculations is at most

$$r \leq \frac{\sqrt{290}/5}{\sqrt{p_0 q_0}} < \frac{3.41}{\sqrt{p_0 q_0}}.$$

By construction, we have $n = p_0 q_0 + s$, with $s \leq p_0 + q_0$. Thus, the approximation ratio is at most

$$\begin{aligned} \text{Rat} &\leq \frac{\frac{\sqrt{290}/5}{\sqrt{p_0 q_0}}}{\sqrt{\frac{1}{\pi + (n-1)(\pi/3 + \sqrt{3}/2)}}} \\ &\leq \frac{\frac{\sqrt{290}/5}{\sqrt{p_0 q_0}}}{\sqrt{\frac{1}{\pi + (p_0 q_0 + p_0 + q_0 - 1)(\pi/3 + \sqrt{3}/2)}}} \\ &= \sqrt{\left(\frac{58\pi}{15} + \frac{29\sqrt{3}}{5}\right) \left(1 + 1/p_0 + 1/q_0\right) + \frac{29(4\pi - 3\sqrt{3})}{15p_0 q_0}} \\ &\approx \sqrt{22.194 \left(1 + 1/p_0 + 1/q_0\right) + \frac{14.249}{p_0 q_0}} \leq \sqrt{22.194 \left(1 + 1/4 + 1/4\right) + \frac{14.249}{4 \cdot 4}} \approx 5.85 < 5.94 \end{aligned}$$

as desired. If we partition Q into a $\lfloor n/q_0 \rfloor \times q_0$ grid (as originally postulated), then the area of each rectangle is

$$\frac{2}{\lfloor n/q_0 \rfloor q_0} \leq \frac{2}{(n/q_0 - 1)q_0} = \frac{2}{n - q_0} \leq \frac{2}{n - \sqrt{n}} < \frac{22}{9n}$$

because we have $q_0^2 \leq n$ and $n \geq 33$.

Our proof is therefore complete if we consider the interval $\sqrt{2n}/4 < w < \frac{3}{5}\sqrt{n}$ and assume that $n \geq 33$. As we explain in Section C of the online supplement, we can achieve the desired approximation ratio by using a grid of dimensions $\lfloor n/3 \rfloor \times 3$ when $\sqrt{2n}/4 \leq w < \sqrt{n/3}$, and a grid of dimensions $n \times 1$ when $\sqrt{n/3} \leq w < \frac{3}{5}\sqrt{n}$, which completes the proof.

5 Computational experiments

In order to compare our three algorithms, this section presents the results of a simple computational experiment using randomly generated input regions C . The following procedure was performed 1000 times:

1. Let n , the desired number of sub-regions, be drawn uniformly between 2 and 60.
2. Let C be the convex hull of m points selected uniformly at random in the unit square, with m also drawn uniformly between 3 and 10 inclusively.
3. Run Algorithm 1, Algorithm 4, and Algorithm 6 on C .
4. For each of the three algorithm outputs, perform the following steps 1000 times:
 - (a) Sample x_1, \dots, x_n uniformly at random from R_1, \dots, R_n .
 - (b) Let r be the connectivity radius of x_1, \dots, x_n .
5. Let r_1 be the largest connectivity radius that was ever obtained in the 1000 samples associated with Algorithm 1.
6. Let r_2 be the largest connectivity radius that was ever obtained in the 1000 samples associated with Algorithm 4.
7. Let r_3 be the largest connectivity radius that was ever obtained in the 1000 samples associated with Algorithm 6.

At the end of this procedure, we have 1000 samples of the form (r_1, r_2, r_3) . Figures 15a and 15b show the ratios r_2/r_1 and r_3/r_1 respectively, as a function of the aspect ratio of C (to be precise, the aspect ratio of the *bounding box* of C). The results are not surprising: first, we see that Algorithm 4 outperforms Algorithm 1 in nearly all cases (since nearly all of the data points have $r_2/r_1 < 1$), with the exception of a small number of samples that occur when the aspect ratio of C is nearly 1. Figure 15b shows that Algorithm 1 may outperform Algorithm 6 somewhat when the aspect ratio of C is very close to 1, but there is again an unmistakable trend in favor of Algorithm 6 as the aspect ratio increases. This is unsurprising because it reflects precisely the phenomenon that motivated us to introduce Algorithm 6 in the first place, namely the “bottleneck” phenomenon that occurs with input shapes that are long and skinny, which is shown in Figure 5.

6 Conclusions

We have presented three approximation algorithms for partitioning a convex region C into sub-regions so as to minimize the connectivity radius of any set of points inside those sub-regions. Our analysis further leads to a fourth algorithm, based on similar principles, that solves the continuous k -centers problem in a convex polygon with approximation factor 1.99. One potential direction for future research would be the imposition of other (weaker) shape constraints on the sub-regions (as opposed to equal area or convexity), such as requiring star convexity or simple connectivity (i.e. no “holes”); structures of such shapes are prevalent in the computational geometry literature [3, 4], and we suspect that efficient approximation algorithms exist for such scenarios as well.

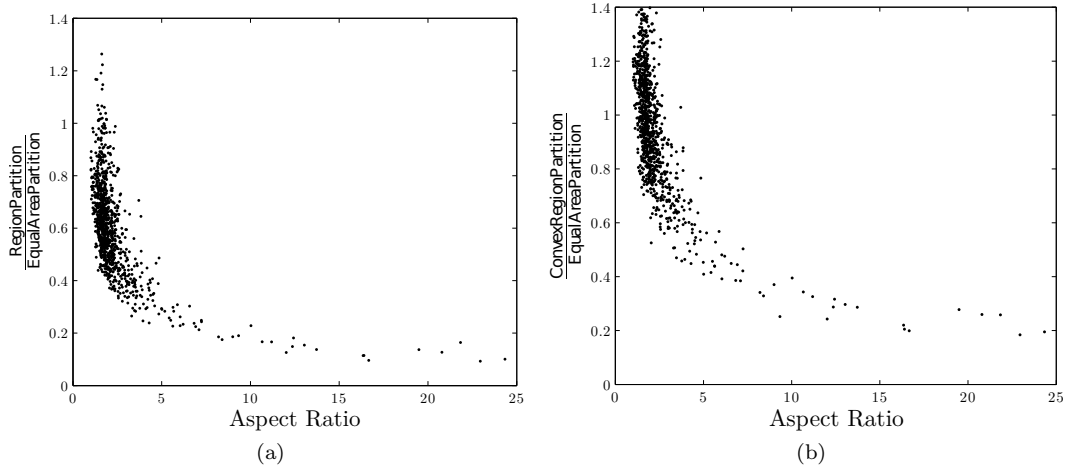


Figure 15: The ratio of the connectivity radii of our three algorithms. Figure 15a shows the ratio of the maximum radius produced by Algorithm 4 to the maximum radius produced by Algorithm 1. Figure 15b shows the same ratios between Algorithms 6 and 1.

References

- [1] K. M. Alzoubi, P.-J. Wan, and O. Frieder. Message-optimal connected dominating sets in mobile ad hoc networks. In *Proceedings of the 3rd ACM international symposium on Mobile ad hoc networking & computing*, pages 157–164. ACM, 2002.
- [2] B. Aronov, P. Carmi, and M.J. Katz. Minimum-cost load-balancing partitions. *Algorithmica*, 54(3):318–336, July 2009.
- [3] D. Avis and G. T. Toussaint. An efficient algorithm for decomposing a polygon into star-shaped polygons. *Pattern Recognition*, 13(6):395–398, 1981.
- [4] S. Bereg, P. Bose, and D. Kirkpatrick. Equitable subdivisions within polygonal regions. *Computational Geometry*, 34(1):20–27, 2006.
- [5] O. Berman, Z. Drezner, A. Tamir, and G. O. Wesolowsky. Optimal location with equitable loads. *Annals of Operations Research*, 167(1):307–325, 2009.
- [6] J. Brimberg and S. Salhi. A continuous location-allocation problem with zone-dependent fixed cost. *Annals of Operations Research*, 136(1):99–115, 2005.
- [7] J.G. Carlsson. Dividing a territory among several vehicles. *INFORMS Journal on Computing*, To appear, 2012.
- [8] J.G. Carlsson, D. Ge, A. Subramaniam, and Y. Ye. Solving the min-max multi-depot vehicle routing problem. In *Proceedings of the FIELDS Workshop on Global Optimization*, 2007.

- [9] D. Haugland, S. C. Ho, and G. Laporte. Designing delivery districts for the vehicle routing problem with stochastic demands. *European Journal of Operational Research*, 180(3):997 – 1010, 2007.
- [10] D. S. Hochbaum. *Approximation algorithms for NP-hard problems*, volume 20. PWS publishing company Boston, 1997.
- [11] J. B. M. Melissen and P. C. Schuur. Covering a rectangle with six and seven circles. *Discrete Applied Mathematics*, 99(1):149–156, 2000.
- [12] Marco Pavone, Alessandro Arsie, Emilio Frazzoli, and Francesco Bullo. Distributed policies for equitable partitioning: theory and applications. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 4191–4197, Piscataway, NJ, USA, 2008. IEEE Press.
- [13] Marco Pavone, Alessandro Arsie, Emilio Frazzoli, and Francesco Bullo. Equitable partitioning policies for robotic networks. In *ICRA'09: Proceedings of the 2009 IEEE international conference on Robotics and Automation*, pages 3979–3984, Piscataway, NJ, USA, 2009. IEEE Press.
- [14] Franco P. Preparata and Michael I. Shamos. *Computational geometry: an introduction*. Springer-Verlag New York, Inc., New York, NY, USA, 1985.
- [15] E. M. Royer, P. M. Melliar-Smith, and L. E. Moser. An analysis of the optimum node density for ad hoc mobile networks. In *Communications, 2001. ICC 2001. IEEE International Conference on*, volume 3, pages 857–861. IEEE, 2001.
- [16] I. Stojmenovic. Position-based routing in ad hoc networks. *Communications Magazine, IEEE*, 40(7):128–134, 2002.
- [17] G. Valiente. *Algorithms on Trees and Graphs*. Springer Berlin Heidelberg, 2013.
- [18] R. Wattenhofer, L. Li, P. Bahl, and Y.-M. Wang. Distributed topology control for power efficient operation in multihop wireless ad hoc networks. In *INFOCOM 2001. Twentieth Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings. IEEE*, volume 3, pages 1388–1397. IEEE, 2001.
- [19] J. Wu and H. Li. On calculating connected dominating set for efficient routing in ad hoc wireless networks. In *Proceedings of the 3rd international workshop on Discrete algorithms and methods for mobile computing and communications*, pages 7–14. ACM, 1999.
- [20] D. C. Youla and H. Webb. Image restoration by the method of convex projections: Part 1: Theory. *Medical Imaging, IEEE Transactions on*, 1(2):81–94, October 1982.

Online supplement to “Geometric partitioning and robust ad-hoc network design”

A Proof of Theorem 12

It remains to consider the special case where $q_0 = 2$; we will decompose this into two further sub-cases in which we will either divide Q into $\lfloor (n-1)/2 \rfloor \times 2$ sub-rectangles or $\lfloor (n-1)/3 \rfloor \times 3$ sub-rectangles. Recall from (3) that we must have

$$\frac{\sqrt{2(n-1)}}{3} < w \leq \frac{\sqrt{2(n-1)}}{2},$$

so that we must have either $\sqrt{2(n-1)}/3 < w \leq \sqrt{2(n-1)}/2.6$ or $\sqrt{2(n-1)}/2.6 < w \leq \sqrt{2(n-1)}/3$:

- If $\sqrt{2(n-1)}/3 < w \leq \sqrt{2(n-1)}/2.6$, then we will decompose Q into $\lfloor (n-1)/3 \rfloor \times 3$ sub-rectangles. By computing vertical and horizontal differences between rectangle centers as before, the approximation ratio is at most

$$\frac{\max \left\{ \frac{2}{3w}, \frac{3w}{(n-1)-3} \right\}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}}.$$

The first term of the $\max\{\cdot, \cdot\}$ expression is largest when w is as small as possible, which occurs when $w = \sqrt{2(n-1)}/3$; the above ratio is then equal to

$$\begin{aligned} \frac{\frac{2}{3w}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} &= \frac{\frac{2}{\sqrt{2(n-1)}}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} \\ &= \frac{\sqrt{(9\sqrt{3}+6\pi)(n-1)+18\pi}}{3\sqrt{n-1}} < 2.77 \text{ for all } n \geq 21 \end{aligned}$$

and the second term of the $\max\{\cdot, \cdot\}$ expression is largest when w is as large as possible, which occurs when $w = \sqrt{2(n-1)}/2.6$; the above ratio is then equal to

$$\begin{aligned} \frac{\frac{3w}{(n-1)-3}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} &= \frac{\frac{3(\sqrt{2(n-1)}/2.6)}{(n-1)-3}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} \\ &< \frac{\sqrt{5.10(n-1)^2+8.37(n-1)}}{n-4} < 2.77 \text{ for all } n \geq 21 \end{aligned}$$

as desired.

- If $\sqrt{2(n-1)}/2.6 < w \leq \sqrt{2(n-1)}/2$, then we will decompose Q into $\lfloor (n-1)/2 \rfloor \times 2$ sub-rectangles. By computing vertical and horizontal differences between rectangle centers as before, the approximation ratio is at most

$$\frac{\max\left\{\frac{1}{w}, \frac{2w}{(n-1)-2}\right\}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}}.$$

The first term of the $\max\{\cdot, \cdot\}$ expression is largest when w is as small as possible, which occurs when $w = \sqrt{2(n-1)}/2.6$; the above ratio is then equal to

$$\begin{aligned} \frac{\frac{1}{w}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} &= \frac{\frac{1}{\sqrt{2(n-1)}/2.6}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} \\ &< \frac{\sqrt{6.47(n-1)} + 10.62}{\sqrt{n-1}} < 2.77 \text{ for all } n \geq 21 \end{aligned}$$

and the second term of the $\max\{\cdot, \cdot\}$ expression is largest when w is as large as possible, which occurs when $w = \sqrt{2(n-1)}/2$; the above ratio is then equal to

$$\begin{aligned} \frac{\frac{2w}{(n-1)-2}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} &= \frac{\frac{\sqrt{2(n-1)}}{(n-1)-2}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} \\ &= \frac{\sqrt{(9\sqrt{3}+6\pi)(n-1)^2 + 18\pi(n-1)}}{3(n-3)} < 2.77 \text{ for all } n \geq 21, \end{aligned}$$

which completes the proof.

B Proof of Theorem 14

If $q_0 = 1$, then we have $1 \leq \sqrt{2k}/w < 2$. We have already seen that our ratio of 1.99 is valid whenever $w \geq \sqrt{k}$, and thus we may restrict ourselves to the domain $\sqrt{2} < \sqrt{2k}/w < 2$, or equivalently $\sqrt{2} < t < 2$ with $t = \sqrt{2k}/w$ as before. The parity of k is now relevant; if k is even, then we can divide Q into a $k/2 \times 2$ grid, wherein each sub-rectangle has dimensions $\frac{w}{k/2} \times h/2$. The distance from any point $x \in C \subseteq Q$ is at most half of the diagonal of such a rectangle, which is

$$\frac{1}{2} \sqrt{\left(\frac{w}{k/2}\right)^2 + \left(\frac{h}{2}\right)^2} = \sqrt{\frac{1}{k}} \cdot \sqrt{\left(\frac{w^2}{k} + \frac{k}{4w^2}\right)} = \sqrt{\frac{1}{k}} \cdot \sqrt{t^2/8 + 2/t^2} < \frac{1}{2} \sqrt{\frac{5}{k}}$$

since $\sqrt{2} < t < 2$. Our approximation ratio is met because

$$\text{Rat} \leq \frac{\frac{1}{2} \sqrt{5/k}}{1/\sqrt{\pi k}} < 1.99.$$

It remains to consider the case where k is odd, which will complete the proof.

Lemma 17. *Let R be a rectangle with dimensions $a \times b$, where $a \geq b$. If 5 points x_1, \dots, x_5 are placed inside R according to Figure 9, then the distance between any point $x \in R$ and its nearest neighbor x_i is at most*

$$\min_i \|x - x_i\| \leq \frac{a}{\pi^2} + \frac{b}{2\varphi},$$

where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio.

Proof. The configurations in Figure 9 are due to [11], which also gives a precise closed form expression for the maximum possible nearest-neighbor distance over all $x \in R$, i.e. $\max_{x \in R} \min_i \|x - x_i\|$. The above inequality is merely a crude upper bound thereof. \square

We will next divide our rectangle Q with a vertical line into two rectangles Q_1 and Q_2 that have dimensions $\ell \times h$ and $(w - \ell) \times h$ respectively, where we let $\ell = 11.08/\sqrt{k} - 6.10/w$, and we will place 5 points in Q_1 and $k - 5$ points in Q_2 . By Lemma 17, the maximum distance between a point $x \in Q_1$ and its nearest neighbor $x_i \in Q_1$ (which is placed according to Figure 9) is

$$\frac{\ell}{\pi^2} + \frac{h}{2\varphi} \approx \frac{1.122638714}{\sqrt{k}} - \frac{0.00003}{w} < 1.1227/\sqrt{k}$$

and thus our desired approximation ratio is met because

$$\frac{1.1227/\sqrt{k}}{1/\sqrt{\pi k}} < 1.99.$$

We can verify that the approximation ratio is met for Q_2 as well, which will complete our entire proof. If we divide Q_2 into a $(k - 5)/2 \times 2$ grid, then each sub-rectangle will have dimensions $\frac{w - \ell}{(k - 5)/2} \times \frac{h}{2}$, and thus using half of the diagonal of such a rectangle is

$$\begin{aligned} \frac{1}{2} \sqrt{\left[\frac{w - \ell}{(k - 5)/2} \right]^2 + \left(\frac{h}{2} \right)^2} &= \frac{\sqrt{\ell^2 - 2w\ell + w^2 + \frac{1}{w^2}(k^2/4 - 5k/2 + 25/4)}}{k - 5} \\ &= \frac{\sqrt{\frac{122.7664}{k} - \frac{135.176}{w\sqrt{k}} - \frac{22.16w}{\sqrt{k}} + w^2 + \frac{1}{w^2}(k^2/4 - 5k/2 + 43.46) + 12.2}}{k - 5} \\ &= \frac{\sqrt{\frac{122.7664}{k} - \frac{67.588\sqrt{2}t}{k} - \frac{22.16\sqrt{2}}{t} + \frac{2k}{t^2} + \frac{kt^2}{8} - \frac{5t^2}{4} + \frac{21.73t^2}{k} + 12.2}}{k - 5} \end{aligned}$$

where we have again substituted $t = \sqrt{2}kw$. The inner term of the square root is convex in t for $k \geq 31$ (by routine algebra) and thus, for fixed k , the above quantity is maximized at $t = \sqrt{2}$ or $t = 2$, and the above expression simply reduces to

$$\frac{1}{k - 5} \max \left\{ \sqrt{\frac{3.10504}{k} + 1.25k - 12.46}, \sqrt{\frac{209.6864 - 135.176\sqrt{2}}{k} + k - 11.08\sqrt{2} + 7.2} \right\}.$$

We therefore merely need to prove that the approximation ratio holds for all $k \geq 31$, i.e. that the ratio of the above expression to the lower bound $1/\sqrt{\pi k}$, which is

$$\begin{aligned}
& \frac{\frac{1}{k-5} \max \left\{ \sqrt{\frac{3.10504}{k} + 1.25k - 12.46}, \sqrt{\frac{209.6864 - 135.176\sqrt{2}}{k} + k - 11.08\sqrt{2} + 7.2} \right\}}{1/\sqrt{\pi k}} \\
&= \frac{1}{k-5} \max \left\{ \sqrt{\pi(3.10504 + 1.25k^2 - 12.46k)}, \sqrt{\pi(209.6864 - 135.176\sqrt{2} + k^2 - (11.08\sqrt{2} - 7.2)k)} \right\} \\
&< \frac{1}{k-5} \max \left\{ \sqrt{3.93k^2 - 39.14k + 9.76}, \sqrt{3.15k^2 - 26.60k + 58.19} \right\}
\end{aligned}$$

is bounded above by 1.99 for all $k \geq 31$. This is a simple univariate function in k and it is routine to verify that the desired result holds, which completes the proof.

C Proof of Theorem 15

Suppose that $\sqrt{2n}/4 \leq w < \sqrt{n/3}$ and we divide Q into a grid of dimensions $p \times q = \lfloor n/3 \rfloor \times 3$. Each grid cell has dimensions $w/p \times h/q = \frac{w}{\lfloor n/3 \rfloor} \times \frac{h}{3}$ and by Lemma 16 the connectivity radius r is at most

$$\begin{aligned}
r &\leq \max \left\{ \sqrt{\left(\frac{2w}{\lfloor n/3 \rfloor} \right)^2 + \left(\frac{h}{3} \right)^2}, \sqrt{\left(\frac{w}{\lfloor n/3 \rfloor} \right)^2 + \left(\frac{2h}{3} \right)^2} \right\} \\
&\leq \max \left\{ \sqrt{\left[\frac{2w}{(n-2)/3} \right]^2 + \left(\frac{h}{3} \right)^2}, \sqrt{\left[\frac{w}{(n-2)/3} \right]^2 + \left(\frac{2h}{3} \right)^2} \right\} \\
&= \sqrt{\max \left\{ \left(\frac{6w}{n-2} \right)^2 + \left(\frac{2}{3w} \right)^2, \left(\frac{3w}{n-2} \right)^2 + \left(\frac{4}{3w} \right)^2 \right\}}
\end{aligned}$$

For fixed values of n , the inner term of the square root is the maximum of two convex functions in w and is therefore convex in w , and is therefore maximized at when w is as large or as small as possible, i.e. at $w = \sqrt{2n}/4$ or at $w = \sqrt{n/3}$. At $w = \sqrt{2n}/4$ we have

$$\sqrt{\max \left\{ \left(\frac{6w}{n-2} \right)^2 + \left(\frac{2}{3w} \right)^2, \left(\frac{3w}{n-2} \right)^2 + \left(\frac{4}{3w} \right)^2 \right\}} \Big|_{w=\sqrt{2n}/4} = \sqrt{\frac{9n}{8(n-2)^2} + \frac{128}{9n}}$$

which gives an approximation ratio bounded by the univariate function

$$\text{Rat} \leq \frac{\sqrt{\frac{9n}{8(n-2)^2} + \frac{128}{9n}}}{\sqrt{\frac{1}{\pi + (n-1)(\pi/3 + \sqrt{3}/2)}}} < 5.94 \text{ for all } n \geq 33$$

and at $w = \sqrt{n/3}$ we have

$$\sqrt{\max \left\{ \left(\frac{6w}{n-2} \right)^2 + \left(\frac{2}{3w} \right)^2, \left(\frac{3w}{n-2} \right)^2 + \left(\frac{4}{3w} \right)^2 \right\}} \Big|_{w=\sqrt{n/3}} = \sqrt{\frac{12n}{(n-2)^2} + \frac{4}{3n}}$$

whence

$$\text{Rat} \leq \frac{\sqrt{\frac{12n}{(n-2)^2} + \frac{4}{3n}}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} < 5.94 \text{ for all } n \geq 33.$$

The area of each sub-region is

$$\frac{2}{3\lfloor n/3 \rfloor} \leq \frac{2}{3(n/3 - 2)} = \frac{2}{n - 6} \leq \frac{22}{9n}$$

since $n \geq 33$. Finally, if $\sqrt{n/3} \leq w < \frac{3}{5}\sqrt{n}$, then (as we have already seen in the very beginning of this proof) the connectivity radius of a grid of dimensions $n \times 1$ is at most $\sqrt{(2w/n)^2 + h^2} = \sqrt{(2w/n)^2 + (2/w)^2}$. For fixed n , this is once again maximized for extreme values of w . At $w = \sqrt{n/3}$ we have

$$\text{Rat} \leq \frac{\sqrt{(2w/n)^2 + (2/w)^2} \Big|_{w=\sqrt{n/3}}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} = \frac{\sqrt{\frac{40}{3n}}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} < 5.94 \text{ for all } n \geq 33$$

and at $w = \frac{3}{5}\sqrt{n}$ we have

$$\text{Rat} \leq \frac{\sqrt{(2w/n)^2 + (2/w)^2} \Big|_{w=\frac{3}{5}\sqrt{n}}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} = \frac{\sqrt{\frac{2824}{225n}}}{\sqrt{\frac{1}{\pi+(n-1)(\pi/3+\sqrt{3}/2)}}} < 5.94 \text{ for all } n \geq 33$$

as desired, which completes the proof (clearly, each sub-region has area equal to $2/n$).