Performance of Number-Between g-Type Statistical Control Charts for Monitoring Adverse Events

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Abstract. Alternate Shewhart-type statistical control charts, called “g” and “h” charts, have been developed for monitoring the number of cases between hospital-acquired infections and other adverse events, such as heart surgery complications, catheter-related infections, surgical site infections, contaminated needle sticks, medication errors, and other care-induced concerns. This article investigates the statistical properties of these new charts and illustrates several design considerations that significantly can improve their operating characteristics and sensitivity, including the use of within-limit rules, a new in-control rule, redefined Bernoulli trials, and probability-based limits. These new charts are based on inverse sampling from geometric and negative binomial distributions, are simple for practitioners to use, and in some cases exhibit significantly greater detection power over conventional binomial-based approaches, particularly for infrequent events and low “defect” rates.

Keywords: SPC, control charts, healthcare, adverse events, geometric distribution, g charts, average run length, OC curves, sensitivity, power

1. Introduction

1.1. Overview of article

This article investigates the performance and operating characteristics of a new type of Shewhart statistical process control (SPC) chart, called “g” and “h” charts, for monitoring the number of cases between hospital-acquired infections or other adverse events. These alternate charts recently were developed and illustrated for monitoring cardiac bypass infections, catheter-related infections, surgical site infections, contaminated needle sticks, medication errors, and other care-induced concerns [3–5]. The national costs of these problems are staggering, including an estimated 770,000 to 2 million injured patients, 44,000 to 180,000 deaths, and an estimated $8.8 billion additional healthcare cost annually [1,2,10,12,14,16,22]. Studies summarized in the recent Institute of Medicine report, To Err is Human, estimated that between 45,000 to 98,000 patients die each year in U.S. hospitals from medical errors, with total national direct and indirect costs between $17 billion to $29 billion [19].

While conventional charts (e.g., $p$, $np$, $u$, and $c$) often result in subgroups being plotted too infrequently for real-time control of these problems, particularly when dealing with infrequent events or low “defect” rates, these new charts are based on inverse sampling to detect process changes or verify improvements faster. Several design considerations can also improve the power of these charts to detect process changes over more conventional methods, including standard within-limit rules, redefined Bernoulli trials, a new in-control rule, and probability-based control limits. Recent examples of the use of these charts include the number of open heart surgeries between post-operative sternum wound infections [25], coronary artery bypass graft (CABG) procedures between adverse events [28], surgeries between surgical site infections [3], days between Clostridium difficile colitis positive stool assays [16], time intervals between occurrences of infectious diseases [19], days between adverse events [27], days between insulin reactions [29], and days between needle sticks [6].

1.2. Events-between g and h control charts

Alternate control charts for such situations, called $g$ and $h$ charts, have been developed previously [3,9,21] for the number of independent Bernoulli trials until the first failure, $X$, and the number until the $n$th failure, $T$, which are geometric and negative binomial random variables, respectively. The probability distribution functions of $X$ and $T$ are

\[
P(X = x) = p(1 - p)^{x-1},
\]

for $x = a, a + 1, a + 2, \ldots, a = 0, 1$

and

\[
P(T = t) = (n(1-a)+t-n-1)\binom{t}{a} (1-p)^{t-na}
\]

for $t = na, na + 1, na + 2, \ldots, a = 0, 1$,

where

\[
T = X_1 + X_2 + \cdots + X_n,
\]

\[
\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n},\quad a = 0, 1
\]
for type II (number-before) and type I (number-until) data, respectively,

\[ E(T) = n \left( \frac{1-p}{p} + a \right), \quad \sigma_T = \sqrt{n(1-p)} \text{ and} \]

\[ E(\bar{X}) = \frac{1-p}{p} + a, \quad \sigma_{\bar{X}} = \sqrt{\frac{1-p}{np^2}}. \]

By substituting the conventional method of maximum likelihood and method of moments estimator for \( p \),

\[ \hat{p} = \frac{1}{\bar{X} - a + 1}, \]

where \( \bar{X} \) is the average of all the individual data in all subgroups (i.e., the average number of procedures, or days between infections or adverse events), corresponding \( k \)-sigma control charts for the parameters-known and parameters-estimated cases can be developed. Alternate calculations based on the minimum variances unbiased estimator

\[ \hat{p} = \frac{1}{\bar{X} - a + 1} \frac{N - 1}{N}, \]

where \( N \) is the total number of infections or adverse events in all \( m \) subgroups used to calculate the control limits, also were developed in Benneyan [3] for starting a chart with very few data. Tables 1 and 2 summarize the resultant upper control limit (UCL), lower control limit (LCL), and center line (CL) in general and for the simplest and most common case for \( n = 1 \) and \( a = 0 \) (i.e., the number between single occurrences), where \( m \) is the number of subgroups used to calculate the control limits.

Note that for cases with LCL = 0, plotting values beneath the LCL is not possible given the non-negative nature of "number-before" data, which translates to the lower control limit having no ability to detect rate increases. A slight variation of this chart with \( a = 1 \), such as for the number of catheters up to and including the next infected catheter, is discussed in [3], although both approaches will yield identical statistical properties. (That is, the geometric type I event of exactly \( X \) cases until the next infection is identical to the geometric type II event of exactly \( X - 1 \) cases before the next infection, so the choice is just a matter of preference.) Also note that alternate approaches, especially given the skewness of geometric data, could use probability-based limits or some other value of \( k \neq 3 \) in order to balance the false alarm rate, detection power, and associated costs. These concepts are explored in greater detail below.

A recent article in this journal demonstrated several applications of these control charts [3]. As two examples, figures 1 and 2 illustrate histograms and \( g \) charts for the number of CABG procedures between post-operative infections and the number of days between positive \( Clostridium difficile colitis \) infected stool assays, with the control limits for conventional \( c \) charts also added for comparison. As can be seen, both data sets exhibit geometrically decaying shapes (with chi-square goodness-of-fit tests producing \( p \) values >> 0.05 in both cases) and appear to exhibit states of statistical control in terms of all points being contained within their control limits.

In figure 2(b), however, note that several within-limit signals such as runs of consecutive values beneath the center-

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**Table 1**

Parameters-known and parameters-estimated \( g \) and \( h \) control chart calculations.

<table>
<thead>
<tr>
<th></th>
<th>Rate known</th>
<th>Rate estimated</th>
<th>MVUE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>g chart: Total number events per subgroup</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper control limit (UCL)</td>
<td>( n \left( \frac{1-p}{p} + a \right) + \sqrt{\frac{n(1-p)}{p^2}} )</td>
<td>( \bar{X} + k \sqrt{n(\bar{X} - a)}(\bar{X} - a + 1) )</td>
<td>( \frac{N}{N-1} \left( \bar{X} + \frac{1-a}{N} \right) + k \sqrt{\frac{1}{n}} ) ( \bar{X} - a + 1 )</td>
</tr>
<tr>
<td>Center line (CL)</td>
<td>( n \left( \frac{1-p}{p} + a \right) )</td>
<td>( \bar{X} )</td>
<td>( \frac{N}{N-1} \left( \bar{X} + \frac{1-a}{N} \right) )</td>
</tr>
<tr>
<td>Lower control limit (LCL)</td>
<td>( n \left( \frac{1-p}{p} + a \right) - \sqrt{\frac{n(1-p)}{p^2}} )</td>
<td>( \bar{X} - k \sqrt{n(\bar{X} - a)}(\bar{X} - a + 1) )</td>
<td>( \frac{N}{N-1} \left( \bar{X} + \frac{1-a}{N} \right) - k \sqrt{\frac{1}{n}} ) ( \bar{X} - a + 1 )</td>
</tr>
</tbody>
</table>

**h chart: Average number events per subgroup**

<table>
<thead>
<tr>
<th></th>
<th>Rate known</th>
<th>Rate estimated</th>
<th>MVUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper control limit (UCL)</td>
<td>( \left( \frac{1-p}{p} + a \right) + k \sqrt{\frac{1-p}{np^2}} )</td>
<td>( \bar{X} + k \sqrt{\frac{\bar{X} - a}{n}}(\bar{X} - a + 1) )</td>
<td>( \frac{N}{N-1} \left( \bar{X} + \frac{1-a}{N} \right) + k \sqrt{\frac{1}{n}} ) ( \bar{X} - a + 1 )</td>
</tr>
<tr>
<td>Center line (CL)</td>
<td>( \left( \frac{1-p}{p} + a \right) )</td>
<td>( \bar{X} )</td>
<td>( \frac{N}{N-1} \left( \bar{X} + \frac{1-a}{N} \right) )</td>
</tr>
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<td>Lower control limit (LCL)</td>
<td>( \left( \frac{1-p}{p} + a \right) - k \sqrt{\frac{1-p}{np^2}} )</td>
<td>( \bar{X} - k \sqrt{\frac{\bar{X} - a}{n}}(\bar{X} - a + 1) )</td>
<td>( \frac{N}{N-1} \left( \bar{X} + \frac{1-a}{N} \right) - k \sqrt{\frac{1}{n}} ) ( \bar{X} - a + 1 )</td>
</tr>
</tbody>
</table>
line [15,17] indicate a rate increase between observations 34–55. Also note that in this example it would be preferable to know the exact number of cases between positive specimens, rather than the number of days, although in many cases such detailed data may not be available easily and the number of days can serve as a reasonable surrogate. In terms of visual interpretation of g charts, note that a decrease in the number of procedures or days between infections corresponds to an increase in the infection rate and to values closer to, rather than farther from, the horizontal x-axis (unlike the case for more familiar types of charts). These charts also are useful in helping test and confirm improvements via values above the upper control limit.

Other recent “number-between” examples of g control charts include the number of procedures between surgical site infections, the number of patients between catheter-associated infections, days between gram stain errors blood cultures taken from patients with pyrexia between blood stream infections (BSI), the number of days between preventable adverse drug events, and the number of intensive care unit patients colonized with *Staphylococcus aureus* between methicillin-resistant (MRSA) cases.

2. Chart performance and statistical properties

2.1. Statistical properties and performance measures

Two important properties of any control chart are the probabilities of detecting true process changes (e.g., infection rate increases) and of incorrectly indicating changes when, in fact, none have occurred. These properties often are summarized graphically by operating characteristic (OC) or average run length (ARL) curves in order to help assess a chart’s statistical characteristics. The calculation and interpretation of these and related measures have been discussed elsewhere [5,15,17], including the expected number of items produced until detection, the medians and standard deviations of run lengths and number produced, and various quantiles of the distributions of these measures. An OC curve is a plot of the probability of any single subgroup not generating an out-of-control signal, on the vertical axis, given any specific in-control or out-of-control values of \( p = p' \) along the horizontal axis; that is

\[
P(\text{no signal } | \ p') = P(\text{LCL} \leq X \leq \text{UCL} | \ p = p') = \sum_{t=\lceil LCL \rceil^+}^{\lceil UCL \rceil^-} P(T = t | p = p') = \sum_{t=\lceil LCL \rceil^+}^{\lceil UCL \rceil^-} nb(t; n, a, p') \]

\[
= \sum_{t=\lceil LCL \rceil^+}^{\lceil UCL \rceil^-} \left( n(1 - a) + t \right) \left( p'^{n} (1 - p')^{t-na} \right)
\]

where \( nb(t; n, a, p') \) is the negative binomial probability distribution function and \( \lceil \cdot \rceil^+ \) and \( \lceil \cdot \rceil^- \) are the integer round-up and round-down functions, respectively. In the frequent case with \( n = 1 \), this also can be expressed in closed form (see appendix) as

\[
P(\text{no signal } | \ p') = \sum_{x=\lceil LCL \rceil^+}^{\lceil UCL \rceil^-} p' (1 - p')^{x-a+1} \]

\[
= \frac{(1 - p')^{\lceil LCL \rceil^+} - (1 - p')^{\lceil UCL \rceil^-}}{(1 - p')^{a-1}}
\]

<p>| Table 2 |
| Calculations for individual adverse event g control charts (( n = 1 )). |</p>
<table>
<thead>
<tr>
<th>Rate known</th>
<th>Rate estimated</th>
<th>MVUE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number-before applications (a = 0)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper control limit (UCL)</td>
<td>( \frac{1-p}{p} + k \sqrt{\frac{1-p}{p^2}} \left[ \frac{\bar{x} + 1}{m} \right] + k \sqrt{\frac{\bar{x} + 1}{m}} (\bar{x} + 1) )</td>
<td>( \frac{m}{m-1} \left[ \frac{\bar{x} + 1}{m} \right] )</td>
</tr>
<tr>
<td>Lower control limit (LCL)</td>
<td>( \frac{1-p}{p} - k \sqrt{\frac{1-p}{p^2}} \left[ \frac{\bar{x} - k}{m} \right] )</td>
<td>( \frac{m}{m-1} \left[ \frac{\bar{x} - k}{m} \right] )</td>
</tr>
<tr>
<td><strong>Number-until applications (a = 1)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper control limit (UCL)</td>
<td>( \frac{1}{p} + k \sqrt{\frac{1-p}{p^2}} \left[ \frac{\bar{x}}{m} \right] + k \sqrt{\frac{\bar{x}}{m}} (\bar{x} - 1) )</td>
<td>( \frac{m}{m-1} \left[ \frac{\bar{x} + 1}{m} \right] )</td>
</tr>
<tr>
<td>Lower control limit (LCL)</td>
<td>( \frac{1}{p} - k \sqrt{\frac{1-p}{p^2}} \left[ \frac{\bar{x} - k}{m} \right] )</td>
<td>( \frac{m}{m-1} \left[ \frac{\bar{x} - k}{m} \right] )</td>
</tr>
</tbody>
</table>

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Figure 1. (a) Comparison of empirical heart surgery infection data with geometric distribution. (b) Heart surgery infection control g chart.
In the standard case for which out-of-control signals are generated only by values outside the control limits, the number of subgroups until an out-of-control signal (i.e., the run length) is itself a geometric random variable with parameter \( r = P(LCL \leq X \leq UCL \mid p = p') \), and therefore OC curves and ARL's are inversely related to each other as

\[
ARL = \frac{1}{\text{probability of signal}}
\]

and thus

\[
\frac{1}{ARL} = P(LCL \leq X \leq UCL \mid p = p') = 1 - \frac{1}{ARL}
\]
In other cases, such as when using within-limit rules, out-of-control probabilities and expected run lengths can be determined via Markov chain or computer simulation analysis. A related performance measure, the average (expected) number of inspected (ANI) items until a signal, is important when comparing different types of charts with different subgroup sizes. The calculations of ARL and ANI for g charts are presented in the appendix. The OC curve shown in figure 3 illustrates the importance of examining a chart’s properties for a scenario similar to that shown earlier in figure 2 with the initial in-control probability of an infection on any given day of \( p \approx 0.35 \).

Note that an ideal curve would rise steeply to a probability of almost 1.0 directly above the in-control value of \( p \) on the x-axis and then descend steeply, corresponding to a low false alarm rate and high true detection power, respectively. This particular \( g \) chart, conversely, exhibits reasonable power to detect rate decreases (i.e., improvements) via the UCL but little-to-no power to detect increases in the infection rate (due to the lower control limit remaining equal to zero), which in many applications often may be the most important type of process change to detect.

2.2. Some design issues and options

Situations with a LCL = 0, such as above, in fact are not uncommon and often occur both for other \( g \) charts and for many other types of charts as well. Traditional \( np \), \( p \), \( c \), \( u \), \( R \), and \( S \) charts all often will have lower control limits equal to zero and, therefore, no capability to detect process changes via the primary rule of values beneath the LCL. In such cases, several design approaches that may improve a \( g \) chart’s operating characteristics include:

- increasing the subgroup size (or reducing the value of \( k \)) to such a point that the LCL > 0;
- using any or all of the “within-limit rules” to detect rate increases;
- using probability-based control limits instead of traditional \( k \)-sigma limits;
- using a simple new out-of-control rule based on a specified number of consecutive points equal to the LCL; or
- reversing the definition of the Bernoulli trial to count the number of “failures” until the first “success” instead of the number of “successes” until the first “failure”.

2.3. Larger subgroups and/or narrower limits

As with other types of charts, in some applications, a simple solution is to form large enough subgroups so that the LCL is greater than the minimum possible value [5,8,9], especially if a sufficient quantity of historical or real-time data are available on a timely basis. For example, setting the standard \( g \) chart parameters-known and parameters-estimated lower control limit calculations to be greater than the minimum possible subgroup total, \( na \), for \( n \geq 1 \) yields

\[
LCL = n\left(\frac{1-p}{p} + a\right) - k\sqrt{n(1-p)} > na
\]

and

\[
LCL = n\bar{x} - k\sqrt{n\bar{x}(\bar{x} - a)(\bar{x} + a + 1)} > na,
\]

respectively. By algebraically rearranging these equations it can be seen that in order to result in a LCL greater than \( na \), the subgroup size \( n \) must be at least

\[
n > \frac{k^2}{(1-p)}
\]
(and thus $n > k^2$ always, as $0 < 1 - p < 1$ by definition) and

$$n > k^2 \left( \frac{\bar{x} - a + 1}{\bar{x} - a} \right)$$

in general and

$$n > k^2 \left( \frac{\bar{x} - a + 1}{\bar{x} - a} \right) \approx \frac{k^2 \bar{x}}{\bar{x} - 1} \approx \frac{k^2 (\bar{x} + 1)}{\bar{x}} > k^2$$

for reasonably large $\bar{x}$ and $a = 0$ or 1.

In the above case where $p \approx 0.35$ and $k = 3$, for example, a minimum subgroup of size

$$n > \frac{3^2}{0.35 - 0.35} = 13.45$$

and thus

$$n \geq 14$$

would be required in order to raise the LCL above the minimum value. Figure 4 illustrates the effect of several different subgroup sizes on detection power for this example; approximate chart performance for other subgroup sizes can be inferred by interpolation. In cases where it is important to interpret every individual adverse event without waiting for a sufficiently large subgroup, a moving average approach of $n$ individual number-between data might be used, with the moving window size set to the above result for $n$.

Alternately, the number of standard deviations, $k$, used in the control limits must be smaller than

$$k < \sqrt{n(1 - p)}$$

(and thus $k < \sqrt{n}$ always, as $0 < 1 - p < 1$ by definition) and

$$k < \sqrt{\frac{n(\bar{x} - a)}{\bar{x} - a + 1}}$$

in general and

$$k < \sqrt{\frac{n(\bar{x} - a)}{\bar{x} - a + 1}} \approx \sqrt{\frac{n \bar{x}}{\bar{x} + 1}} \approx \sqrt{\frac{n(\bar{x} - 1)}{\bar{x}}} < \sqrt{n}$$

for reasonably large $\bar{x}$ and $a = 0$ or $a = 1$, for the LCL to have detection power, both implying that $k < 1$ in the case of individual measurements with $n = 1$. Note that control limits this narrow should almost never be recommended and thus are not really an option, as this would unacceptably increase the false alarm rate, as shown in figure 5 by the OC curves corresponding to $k = 0.5$ and 1. Also note that the above inequalities imply for the LCL to have any power then the infection rate, $p$, or the average number of cases between infections, $\bar{x}$, mathematically must be

$$p < \frac{n - k^2}{n}$$

or

$$\bar{x} - a < -\frac{k^2}{k^2 - n} \quad \text{(for } n < k^2),$$

respectively. As can be seen by examination, both these conditions result in negative values of $p$ when $n = 1$ and $k > 1$, and therefore a non-zero LCL is not possible for any infection rate under these common conditions. The condition that $k < 1$ for LCL > 0 also can be seen to make sense by noting that the mean and standard deviation are equal for any exponential random variable, the continuous analogue to a geometric random variable.
2.4. Use of within-limit supplementary rules

As an alternate approach, another option is to use standard "within-limit" supplementary rules for detecting unnatural variability between the control limits. For example, the use of these rules for $\bar{X}$ charts has been shown to significantly reduce average detection run lengths (although with moderate increases in false alarms) [13] and their improvement of $np$, $c$, and $g$ charts for various subgroup sizes and nonconforming rates has been explored [5,7,9] in detail via Markov and simulation analysis.

In the above example (with $p \approx 0.35$), figure 6 illustrates the average run lengths of the corresponding $g$ chart both with and without supplementary rules, for the two subgroup sizes $n = 1$ and 5. As shown, these additional out-of-control criteria improve the ability to detect rate increases, with larger improvement for larger subgroup sizes, as is intuitive, and with slight increases in false alarms. (Note that the "without rules" ARL curves correspond to the OC curves shown previously in figure 3.) Again note that larger subgroup sizes, if possible, result in better operating characteristics both with and without supplementary rules.
2.5. A new lower control limit rule

A related option for detecting rate increases in cases with LCL = na (i.e., the minimum possible value) is to develop a simple new supplementary rule based on a number of consecutive values, \( M \), equal to the lower control limit. For example, if \( \alpha_{LCL,M} \) is the desired LCL “M-sequence” false alarm rate associated with a sequence of \( M \) consecutive independent values, in the sense that all \( M \) values will equal the LCL with probability \( \alpha_{LCL,M} \) given an in-control rate \( p_o \), then

\[
\alpha_{LCL,M} = P(X_1 = X_2 = \cdots = X_M = na \mid p_o)
= [P(X = na \mid p_o)]^M
= \left[ \left( \frac{n(1-a) + na - 1}{n-1} \right) p_o^n (1-p_o)^{na-na} \right]^M
= \left[ \left( \frac{n-1}{n-1} \right) p_o^n (1-p_o)^0 \right]^M = p_o^{nM},
\]

which for the common case with \( n = 1 \) further simplifies to \( \alpha_{LCL,M} = p_o^M \). Solving for an integer value of \( M \) to attain a probability as close as possible to, but not larger than, \( \alpha_{LCL,M} \) produces

\[
M = \left[ \frac{\ln \alpha_{LCL,M}}{n \ln p_o} \right]^+ \quad \text{for } n \geq 1
\]

and

\[
M = \left[ \ln p_o \right]^+ \quad \text{for } n = 1,
\]

where \([ ]^+\) indicates the round-up function to increase non-integer results to the next integer. Note that the true \( \alpha_{LCL,M} \)
value will be slightly less than the desired rate \( \alpha_{LCL,M} \) due to the non-integer solution to the above, with

\[
\text{true } \alpha_{LCL,M} = p_o^{n[M^+]} = p_o^{n[\ln \alpha_{LCL,M} / (n \ln p_o)]^+}.
\]

Similarly, the probability of \( M \) constant values on the LCL if \( p_o \) shifts to another value \( p_a \) is

\[
P(\text{signal rate increase } \mid p_o) = 1 - \beta_{LCL,M}
= p_a^n[M^+] = p_a^n[\ln \alpha_{LCL,M} / (n \ln p_o)]^+.
\]

These results can be used to determine the values of \( M \) and \( n \) necessary to achieve preferred \( \alpha_{LCL,M} \) and \( \beta_{LCL,M} \) values, with lower values of the in-control rate \( p_o \) allowing more precise attainment of desired properties. In the present case with \( p_o \approx 0.35 \) and \( n = 1 \), for example, using a desired \( \alpha_{LCL,M} = 0.01 \) then the above yields

\[
M = \left[ \frac{1}{(1 \ln 0.35)} \right]^+ \approx [4.387]^+ = 5,
\]

and if the process remains in statistical control then this in turn results in an actual false alarm probability of

\[
\text{true } \alpha_{LCL,M} = 0.35^{[\ln 0.01 / \ln 0.35]^+} = 0.35^5 \approx 0.00525.
\]

Conversely, if the infection rate increased to \( p_a = 0.70 \), then the probability that a sequence of \( M = 5 \) consecutive values detects this increase is

\[
P(\text{sequence of 5 points detect } \mid p_a) = 1 - \beta_{LCL,M,p_a}
= 0.70^{[\ln 0.01 / \ln 0.35]^+} = 0.70^5 \approx 0.16807.
\]

Figure 7 summarizes the probability of an out-of-control signal for this example across all possible shifts in the infection rate, where the y-axis here represents the probability that
any independent sequence of \( M \) points together will generate a signal. While these results can not be directly equated to OC curves and ARL’s for independent points (e.g., due to the lack of independence between immediately overlapping sequences of subgroups, such as \((X_i, X_{i+1}, \ldots, X_{i+M})\) and \((X_{i+1}, X_{i+2}, \ldots, X_{i+M+1})\)), the relative improvement in performance still is clear. Although not the current focus here, moreover, exact average run lengths could be studied as discussed above.

### 2.6. Counting “successes” versus “failures”

A fourth interesting option that often will increase detection power is to conceptually reverse the definition of “success” and “failure” of the Bernoulli trials (i.e., procedures, surgeries, days, catheters, etcetera), for example, such as instead to count the number of consecutive cases with infections between cases without infections. Although in the binomial case counting “successes” instead of “failures” will produce control charts with exactly the same OC curves, it is interesting that this is not the case for geometric events and, in fact, can significantly alter the chart’s statistical properties.

As an illustration, figure 8 compares the difference in OC curves for the earlier \( g \) chart with \( p \approx 0.35 \) (based on the number of days without infections between days with infections) with the “inverse” \( g' \) chart constructed now based on the number of days with infections between days without infections, here using \( p' = q = 1 - 0.35 = 0.65 \). In direct contrast to the original \( g \) chart, note that this \( g' \) chart now has relatively decent power to detect rate increases but little-to-no power to detect decreases, and with points closer to the LCL now representing better, not worse, outcomes. In fact, using a \( g' \) versus \( g \) chart can be shown to exactly (for \( n = 1 \)) or nearly exactly (for \( n > 1 \)) reverse the OC curve symmetrically about \( p_o = 0.50 \) to a mirror image of itself. Although not the primary focus here, this interesting result suggests the use of some type of combined \( g \) and \( g' \) SPC scheme (perhaps with the control limits widened appropriately to control the overall false alarm rate in a Bonferroni-type sense).

Also note that it may be beneficial to intentionally define the Bernoulli trial as the number-until-first-failure versus the number-until-first-success such that the associated rate (i.e., the probability of a “failure” or a “success”) is less than 0.5 (or equivalently, so that \( \bar{x} > 1 + \alpha \)), in order to minimize the subgroup size necessary for \( \text{LCL} > na \), as discussed previously, in order to ensure control limit power to detect rate changes in either direction. For example, in the present application, the inverse \( g' \) chart has a rate of \( q = 1 - p \approx 0.65 \) and thus the minimum subgroup size for the \( k = 3 \) sigma \( g' \) chart to have \( \text{LCL} > na = 0 \) is

\[
 n_g' \geq \left[ \frac{k^2}{1 - p'_g} \right]^{1/\alpha} \approx \left[ \frac{3}{\sqrt{1 - 0.65}} \right]^{1/\alpha} \approx [5.071]^{1/\alpha} = 6,
\]

whereas recall that previously the minimum subgroup size for the original \( k = 3 \) sigma \( g \) chart was smaller, with

\[
 n_g \geq \left[ \frac{k^2}{1 - p_g} \right]^{1/\alpha} \approx \left[ \frac{3}{\sqrt{1 - 0.35}} \right]^{1/\alpha} \approx [3.721]^{1/\alpha} = 4.
\]

### 2.7. Probability-based limits

A final chart design option that is relatively simple for practitioners to implement and that is especially useful for low rates is to use “probability-based” control limits instead of the more familiar \( k \)-sigma type limits. In the common case with \( n = 1 \), for example, if \( \alpha_{LCL} \) and \( \alpha_{UCL} \) are the desired (maximum) false alarm rates for the lower and upper control limits, respectively, then appropriate \( g \) chart probability-based control limits can be calculated as follows. (Note that in many cases by convention one might set \( \alpha_{UCL} = \alpha_{LCL} = \alpha_T/2 \), where \( \alpha_T \) is the overall total desired false alarm rate.

![Figure 8. Comparison of “cases between events” versus “cases between non-events”](image-url)
with equal false alarm probabilities for each control limit, although this must not necessarily the case.)

The geometric cumulative distribution function for the integers \( x = a, a + 1, a + 2, \ldots \) can be expressed in closed form as

\[
P(X \leq x) = \sum_{i=a}^{x} (1 - p)^{i-a} = p \sum_{i=0}^{x-a-1} (1 - p)^i = 1 - (1 - p)^{x-a-1}, \quad x = a, a + 1, a + 2, \ldots,
\]

by using the finite series

\[
\sum_{j=b}^{c} d^j = \frac{d^b - d^{c+1}}{1 - d}, \quad \text{for } |d| < 1,
\]

where here \( b = 0, c = x \), and \( 0 < d = 1 - p < 1 \) by definition. Thus for integer and non-integer \( x \geq a \), the probabilities of being strictly less than some value \( x \)

\[
P(X < x) = P(X \leq x - 1) = 1 - (1 - p)^{x-a}
\]

and

\[
P(X < x) = P(X \leq [x]^+ - 1) = 1 - (1 - p)^{[x]^+ - a},
\]

respectively, with the probability of the random variable \( X \) being less than the lower control limit being

\[
P(X < LCL) = 1 - (1 - p)^{LCL-a}, \quad \text{for integer, } LCL = a, a + 1,
\]

\[
P(X < LCL) = 1 - (1 - p)^{LCL^+ - a}, \quad \text{for } LCL \geq a,
\]

\[
\text{LCL not necessarily integer.}
\]

Similarly, the probability of exceeding the upper control limit becomes

\[
P(X > UCL) = 1 - P(X \leq UCL) = 1 - P(X \leq [UCL]^+) = 1 - [1 - (1 - p)^{UCL^- - a+1}],
\]

where \( \lfloor \cdot \rfloor^+ \) indicates the round-down function, and thus

\[
P(X > UCL) = \begin{cases} (1 - p)^{UCL-a+1}, & \text{for integer } UCL, \\ (1 - p)^{UCL^- - a+1}, & \text{for } UCL \text{ not necessarily integer.}
\end{cases}
\]

Setting these results equal to the desired false alarm probabilities \( \alpha_{LCL} \) and \( \alpha_{UCL} \),

\[
\alpha_{LCL} = P(X < LCL) = 1 - (1 - p)^{LCL-a}
\]

\[
\alpha_{UCL} = P(X > UCL) = (1 - p)^{UCL-a+1},
\]

and then solving for the control limits yields

\[
\text{LCL} = \frac{\ln(1 - \alpha_{LCL})}{\ln(1 - p)} + a = \left[ \frac{\ln(1 - \alpha_{LCL})}{\ln(1 - p)} + a \right]^+
\]

\[
\text{UCL} = \frac{\ln(\alpha_{UCL})}{\ln(1 - p)} + a - 1 = \left[ \frac{\ln(\alpha_{UCL})}{\ln(1 - p)} + a - 1 \right]^+
\]

Similarly, the center line can be set to the theoretic median so that the probability of being less than the CL or alternatively less than or equal to the CL is

\[
P(X < CL) = 1 - (1 - p)^{CL-a} \equiv 0.5
\]

or alternatively

\[
P(X \leq CL) = 1 - (1 - p)^{CL-a+1} \equiv 0.5,
\]

respectively, which solve for

\[
\text{CL} = \left[ \frac{\ln 0.5}{\ln(1 - p)} + a \right]^-
\]

and

\[
\text{CL} = \left[ \frac{\ln 0.5}{\ln(1 - p)} + a - 1 \right]^-
\]

respectively (the rounding can be omitted in the CL case).

Table 3 summarizes the control limit and center line calculations for probability-based \( g \) charts when \( n = 1 \). Note that the resultant values for the true \( \alpha_{LCL} \) and \( \alpha_{UCL} \) typically will

<table>
<thead>
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<td>Probability-based ( g ) control charts (( n = 1 )).</td>
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<table>
<thead>
<tr>
<th>Rate known</th>
<th>Rate estimated</th>
<th>MVUE</th>
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<tbody>
<tr>
<td><strong>Upper control limit (UCL)</strong></td>
<td>[ \left[ \frac{\ln(\alpha_{LCL})}{\ln(1 - p)} + a - 1 \right]^+ ]</td>
<td>[ \left[ \frac{\ln(\alpha_{UCL})}{\ln(1 - p)} + a - 1 \right]^+ ]</td>
</tr>
<tr>
<td><strong>Center line (CL)</strong></td>
<td>[ \left[ \frac{\ln 0.5 + a}{\ln(1 - p)} \right]^+ ]</td>
<td>[ \left[ \frac{\ln 0.5 + a}{\ln(1 - p)} \right]^+ ]</td>
</tr>
<tr>
<td><strong>Lower control limit (LCL)</strong></td>
<td>[ \left[ \frac{\ln(1 - \alpha_{LCL})}{\ln(1 - p)} + a \right]^- ]</td>
<td>[ \left[ \frac{\ln(1 - \alpha_{LCL})}{\ln(1 - p)} + a \right]^+ ]</td>
</tr>
</tbody>
</table>
be less than the desired values due to the initial non-integer solution to the above, with

$$P(X < \text{LCL})$$

true \( \alpha_{LCL} \) \hspace{1cm} (1 - p)^{[(\ln(1-\alpha_{LCL})/\ln(1-p))^{+}]} + p^{[(\ln(1-\alpha_{LCL})/\ln(1-p))^{+}]} \]

true \( \alpha_{UCL} \) \hspace{1cm} P(X > \text{UCL})$$

$$= (1 - p)^{[(\ln(1-\alpha_{UCL})/\ln(1-p))^{+}]} + p^{[(\ln(1-\alpha_{UCL})/\ln(1-p))^{+}]}$$

and the true probability of being less than the CL will be

$$P(X < \text{CL})$$

$$= (1 - p)^{[(\ln(0.5)/\ln(1-p))^{+}]} + p^{[(\ln(0.5)/\ln(1-p))^{+}]}$$

The power of each limit to detect rate changes also can be examined by substituting \( p_o \) and \( p_a \) for \( p \) such that

$$\beta_{LCL} = P(X < \text{LCL})$$

$$= (1 - p_a)^{[(\ln(1-\alpha_{LCL})/\ln(1-p_a))^{+}]} + p_a^{[(\ln(1-\alpha_{LCL})/\ln(1-p_a))^{+}]}$$

and

$$\beta_{UCL} = P(X > \text{UCL})$$

$$= (1 - p_a)^{[(\ln(1-\alpha_{UCL})/\ln(1-p_a))^{+}]} + p_a^{[(\ln(1-\alpha_{UCL})/\ln(1-p_a))^{+}]}$$

where \( p_o \) is the original in-control rate (upon which the limits are based) and \( p_a \) is the out-of-control rate of concern to detect. Although not the primary focus here, probability limits similarly also could be developed recursively for the negative binomial g chart case with \( n > 1 \). As an illustration for the common case in which \( n = 1 \), in the earlier CABG heart surgery infection example with \( p_o \approx 0.0237 \) and \( a = 0 \), setting the desired false alarm rates to \( \alpha_{LCL} = \alpha_{UCL} = 0.025 \) yields

$$\text{LCL} \quad \frac{\ln(1 - 0.025)}{\ln(1 - 0.0237)} + 0 \approx 1.0556$$

or

$$\text{LCL} \quad \left[ \frac{\ln(1 - 0.025)}{\ln(1 - 0.0237)} + 0 \right]$$

$$\text{UCL} \quad \frac{\ln(0.025)}{\ln(1 - 0.0237)} + 0 \approx 152.7971$$

$$\text{UCL} \quad \left[ \frac{\ln(0.025)}{\ln(1 - 0.0237)} 0 \right] 153.$$ and

$$\text{CL} \quad \frac{\ln(0.5)}{\ln(1 - 0.0237)} + 0 \approx 28.8988$$

$$\text{CL} = \left[ \frac{\ln(0.5)}{\ln(1 - 0.0237)} + 0 \right] = 28,$$ which if the process remains in statistical control at \( p \approx 0.0237 \) results in

true \( \alpha_{LCL} \)

$$= (1 - 0.0237)^{[(\ln(1-0.0237)/\ln(1-0.025))]} + 0.025 \leq \alpha_{LCL} \approx 0.0237,$$

true \( \alpha_{UCL} \)

$$= (1 - 0.0237)^{[(\ln(0.025)/\ln(1-0.0237))]} + 0.0249 \leq \alpha_{UCL} = 0.025$$

and

$$P(X < \text{CL})$$

$$= (1 - 0.0237)^{[(\ln(0.5)/\ln(1-0.0237))]} + 0.0249 \leq \alpha_{UCL} = 0.025$$

Note that the lower the value of the in-control rate \( p = p_o \), the closer the true \( \alpha_{LCL} \) and \( \alpha_{UCL} \) values will be to their desired values. (Also note that in order to be able to detect rate increases, probability-limits are useful mostly for initial in-control rates less than the desired lower control limit false alarm probability, \( \alpha_{LCL} \), otherwise the true \( \alpha_{LCL} = 0 \).) As an illustration, in the above example if the rate increases from \( p_o = 0.0237 \) to \( p_o = 0.05 \), then

$$\beta_{LCL} = (1 - 0.05)^{1} \quad 0.05$$

and

$$\beta_{UCL} \quad (1 - 0.05)^{154} \approx 0.000371,$$

which equates to \( \text{ARL}_{LCL} = 20 \) and \( \text{ARL}_{UCL} \approx 2695 \), respectively, whereas if the rate decreases to \( p_o = 0.01 \), then

$$\beta_{LCL} \quad 1 \quad (1 - 0.01) \quad 0.01 \quad \text{(or ARL}_{LCL} \quad 100)$$

and

$$\beta_{UCL} \quad (1 - 0.01)^{154} \approx 0.2127 \quad \text{(or ARL}_{UCL} \quad 4.70)$$

Figure 9(a) summarizes these operating characteristics for all possible values of \( p_o \) for the in-control rate \( p_o = 0.0237 \), as compared to conventional \( k = 3 \) standard deviation symmetric control limits and \( k = z_a = 0.05 = 1.96 \) standard deviation symmetric warning limits, with \( n = 1 \) in each case. As shown, the \( \alpha/2 = 0.025 \) equal probability limits (with \( \alpha = 0.05 \)) tend to result in better operating characteristics, in terms of power to detect rate changes in both directions, than the \( z_a = 0.05 \) \( k \)-sigma symmetric limits. In particular, for the case with \( n = 1 \) note the improvement of the \( \alpha = 0.05 \) probability limits over the \( z_a = 0.05 \) \( k \)-sigma limits, which have essentially the same false alarm rate but no power to detect rate increases. Also note that for the \( k = 3 \) and 1.96 cases, minimum subgroup sizes of

$$n > \frac{k^2}{1 - p} = \frac{3^2}{1 - 0.0237} \quad 9.28 \quad n \geq 10.$$
and

\[
\frac{1.96^2}{1 - 0.0237} \approx 3.93 \Rightarrow n \geq 4,
\]

respectively, would be required as discussed earlier in order for the LCL to have any detection power. Figure 9(b), therefore, compares the above OC curves with those corresponding to these situations \((k = 1.96 \text{ and } n = 4, k = 3 \text{ and } n = 10)\) and to their \(\alpha = 0.05\) counterparts \((\alpha = 0.05 \text{ and } n = 4, \alpha = 0.05 \text{ and } n = 10)\). As shown, in each case the probability limits still result in better sensitivity and OC curves, even with the \(k\)-sigma LCL > \(na\).

2.8. Comparison to binomial-based \(np\) and \(p\) charts

It also is interesting to compare the performance of \(g\) and \(np\) control charts when the underlying process is a series of Bernoulli trials and in which a practitioner, knowingly or not, chooses to analyze Bernoulli trials in the context of either geometric or binomial random variables, respectively. Figures 10–12 compare the average number of individual items (Bernoulli trials) until an out-of-control signal is generated by \(g\), \(g'\), and \(np\) charts, here each based on conventional 3-sigma limits. In each case, the subgroup size for the \(g\) and \(g'\) charts is \(n = 1\) and for the \(np\) charts is deter-
Figure 10. Comparison of \( g \) and \( np \) charts: average number Bernoulli trials until signal.

mined by using the conventional rule-of-thumb to choose \( n \) so that \( np \geq 5 \).

Figure 10 compares the average number of items (ANI) – i.e., Bernoulli trials – until an out-of-control signal is generated for an initial in-control rate of \( p = 0.0237 \) if the Bernoulli trials are aggregated into either geometric or binomial events and examined via a \( g \) or \( np \) chart, respectively. The results for \( np \) charts were computed in a similar manner as developed in the appendix for \( g \) charts, now using the binomial probability distribution. As this figure illustrates, note that while the \( g \) chart is ineffective for detecting rate increases, the \( np \) chart is ineffective for detecting rate decreases, with each chart again in a sense compensating for the strengths and deficiencies of the other.

In order to compare the relative performance of each chart for scenarios with other in-control rates of \( p \), figures 11(a) and (b) illustrates the ANI of \( k = 3 \) sigma \( g \), \( g' \), and \( np \) charts for increases by 25% and decreases by 25%, respectively, in the in-control rate of \( p \) for all possible initial values of \( p_0 \) (plotted along the \( x \)-axis). Note that in all cases either a \( g \) chart or a \( g' \) chart yields better performance in terms of better sensitivity than an \( np \) chart (\( g \) chart for decreases, \( g' \) chart for increases). The \( g \) chart ANI when \( p \) increases is not infinity, even though LCL = 0, due to the small but non-zero probability of a value above the UCL (and similarly for \( p \) and \( g' \) charts when \( p \) decreases). Of course, slightly different results may be obtained for other magnitudes of process shifts, for other subgroup size rules-of-thumb, and if probability limits are used instead of \( k \)-sigma limits. For example, figure 12 illustrates the same comparison now based on \( \alpha = 0.005 \) probability limits, again resulting in a \( g \) chart or a \( g' \) chart being superior to \( np \) charts for detecting rate decreases (\( g \)) or increases (\( g' \)).

Other preliminary comparisons also suggest similar advantages in cases with low rates and immediate availability of each observation (also see Bourke [11], Lucas [23], Xie and Goh [31]). As was the case previously in comparing \( g \) and \( g' \) charts, both of these examples again suggest opportunities to explore combined \( g \) and \( np \) SPC schemes so as to somehow combine the best operating characteristics of each chart. However, note that some attention may be necessary so as to not produce "biased" operating characteristics in the sense of greater run lengths occurring for out-of-control rather than in-control states. This type of scenario is most evident in figure 10, with an increase in \( p \) for its in-control value of \( p_0 = 0.0237 \) to an out-of-control value of approximately \( p_u = 0.032 \) actually increasing, not decreasing, by over 350% the average number of surgeries until detection.

3. Discussion

This article explored the performance of several approaches to designing geometric-based SPC charts, called \( g \) and \( h \) charts, for monitoring nosocomial infections and other adverse events as processes over time. Because the costs of these problems can be quite high, rapid detection of a rate increase is of obvious interest, as well as decreases so that root causes can be investigated and standardized). An appealing surveillance feature of these new charts is that they can take immediate advantage of each observed adverse event, rather than waiting until the end of a pre-specified time period, increasing the likelihood of identifying root causes soon after detecting rate changes. These charts also are particularly useful in process improvement work for verifying improvements (via values above the UCL).

As this article illustrates, several alternatives to conventional \( k \)-sigma limits can increase the power of \( g \) charts to detect rate changes, although care should be taken not to overly increase false alarm rates. Interestingly, in some cases advantages may exist to track the number of "successes" between "failures" instead of the number of "failures" between "successes" (unlike the binomial case for which counting
Figure 11. (a) Average number Bernoulli trials to detect increases in \( p \) by 25% \((k = 3\) sigma limits). (b) Average number Bernoulli trials to detect decreases in \( p \) by 25% \((k = 3\) sigma limits).

"successes" or "failures" makes no difference. Of course, the exact impact will vary dependent upon the initial in-control rate, and various combinations of these approaches also might be investigated to achieve the most desirable properties. Also note that while larger subgroups result in higher power to detect smaller shifts, certain uses of number-between data often constrains the subgroup size to \( n = 1 \). Conversely, in cases where periodic single outliers can occur and the process then self-correct, use of the number until the \( n \)th adverse event or a moving average approach might be beneficial to detect only sustained shifts and to reduce false alarms based on single outliers.
Figure 12. (a) Average number Bernoulli trials to detect increases in \( p \) by 25\% (\( \alpha = 0.005 \) probability limits). (b) Average number Bernoulli trials to detect 25\% decreases in \( p \) by 25\% (\( \alpha = 0.005 \) probability limits).

An alternative approach to applying SPC to numbers between data might be to search for a transformation to make empirical data appear reasonably symmetric. Although situations exist where normalizing and other transformations can be advocated [26], in general blind transformation without statistical basis and prior knowledge of process stability can be problematic technically and can reduce the ability to identify existent unnatural variability, a fundamental purpose of SPC. As an example, an inappropriate transformation recently resulted in falsely reporting process stability when the nosocomial infection rate in fact had significantly increased [16]. Also note that most such approaches often lead to the less-preferable and problematic individuals/moving range type of approach and will not at all change
the chart’s statistical operating properties, but only re-shape the probability space to be more symmetric and visually appealing. Nonparametric, pseudo-nonparametric, and “robust” SPC approaches also may be possible, having the advantage of not being as reliant on the exact underlying distribution.

Finally, mathematical methods exist for determining the optimal subgroup size, sampling period, and control limit width in any particular situation. Typically, the subgroup size and sampling period are more affected than the limit width [5], with larger subgroups taken more frequently as the relative cost of a process change increases, which agrees with intuition. As noted previously, however, when plotting every occurrence of infrequent adverse events the sampling frequency and subgroup size are no longer flexible decision variables (all data are plotted, with n = 1), and further investigation into the economic design of g control charts therefore is warranted. Related work might include alternate estimation methods, especially for cases with very little data and start-up conditions, including minimum variance unbiased estimators and geometric-based versions of the recently proposed “Q” charts [30] (also see Bourke [11] and Hawkins [18]). These charts in essence are based on Rao-Blackwell type uniform minimum variance unbiased distribution estimating function and subsequent inverse CDF transformations to achieve approximately i.i.d. normal(0,1) variates, especially useful for start-up and other scenarios where insufficient data exist to use the conventional “estimate-and-insert” approach to the parameters-unknown case.

Appendix

The power and false alarm rate for g and h charts in the case where n = 1 can be calculated as follows. Using the results that the geometric cumulative distribution function is

\[ G(x; a, p) = 1 - (1 - p)^{[x] + a} \]

and that

\[ P(X < x) = 1 - (1 - p)^{[x] + a} \] (see section 2.7), then for any given in-contro or out-of-control value of p = p' the

\[
P(LCL \leq X \leq UCL | p = p') = P(X \leq UCL) - P(X \leq LCL - 1^+)^*
\]

\[
= [1 - (1 - p')^{[UCL] + a - 1}]
\]

\[
- [1 - (1 - p')^{[LCL - 1] + a - 1}]
\]

\[
= (1 - p')^{[LCL - 1]^+} - (1 - p')^{[UCL] - a - 1} - (1 - p')^{[UCL] - a - 1} - (1 - p')^{[LCL - 1]^+} - (1 - p')^{[UCL] - a - 1}
\]

or equivalently

\[
P(LCL \leq X \leq UCL | p = p')
\]

\[
= 1 - [P(X < LCL) + P(X > UCL)]
\]

\[
= 1 - [1 - (1 - p')^{[LCL] + a}]
\]

\[
+ [(1 - p')^{[UCL] - a - 1}]
\]

\[
= (1 - p')^{[LCL] + (1 - p')^{[LCL] - a}}
\]

\[
- (1 - p')^{[UCL + 1]^+} - (1 - p')^{[LCL] - a - 1}
\]

\[
= (1 - p')^{[LCL] + (1 - p')^{[LCL] - a}}
\]

\[
- (1 - p')^{[UCL + 1]^+} - (1 - p')^{[LCL] - a - 1}
\]

\[
= (1 - p')^{[LCL] + (1 - p')^{[LCL] - a}}
\]

\[
- (1 - p')^{[UCL + 1]^+} - (1 - p')^{[LCL] - a - 1}
\]

for LCL ≥ a, UCL ≥ a and where LCL and UCL are derived based on the in-control value of p. The OC curve then can be developed by iterating across values of p’ from 0 to 1.

The average run length (ARL) until an out-of-control signal for a g chart for T or X with any size n then becomes

\[
ARL = \frac{1}{P(T < LCL) + P(T > UCL)}
\]

\[
= \frac{1}{1 - P(LCL \leq T \leq UCL)}
\]

\[
= \left[ \sum_{t=n}^{[UCL] - 1} \left( \frac{n(1 - a) + t - 1}{n - 1} \right) (p')^n (1 - p')^{t - na} \right]^{-1}
\]

\[
+ 1 - \sum_{t=n}^{[UCL] - 1} \left( \frac{n(1 - a) + t - 1}{n - 1} \right)
\]

\[
\times (p')^n (1 - p')^{t - na}
\]

\[
\left[ \sum_{t=max[na, (LCL)]}^{[UCL] - 1} \left( \frac{n(1 - a) + t - 1}{n - 1} \right) \right]^{-1}
\]

\[
\times (p')^n (1 - p')^{t - na}
\]

in general and

\[
ARL = \frac{1}{P(X < LCL) + P(X > UCL)}
\]

\[
= \frac{1}{1 - P(LCL \leq X \leq UCL)}
\]

\[
= \left[ 1 - \frac{(1 - p')^{[LCL] + a - 1} - (1 - p')^{[UCL] - a - 1}}{(1 - p')^{a - 1}} \right]
\]

for X when n = 1.

The average number inspected (ANI) until an out-of-control signal then is equal to the expected number of subgroups until a signal multiplied by the expected number of items in each subgroup.

\[
ANI = ARL \times E(T)
\]

\[
= \left[ 1 - \sum_{t = max[na, (LCL)]}^{[UCL] - 1} \left( \frac{n(1 - a) + t - 1}{n - 1} \right) \right]^{-1} \left[ n \left( \frac{1 - p'}{p' + a} \right) \right]
\]
in general and
\[
\text{ANI} = \frac{(1 - p)^{a-1} - (1 - p)(\text{LCL}-1)^{a} - (1 - p)(\text{UCL})^{a}}{(1 - p')^{a-1}} \times \frac{1-p'}{p'} + a
\]
for the case with \( n = \).

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