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Identificacion de Sistemas Estructurales y su Uso en la Ingeneria Estructural Moderna

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IDENTIFICACION

Formulación en Espacio de Estado

En la premisa de que el amortiguamiento es viscoso las ecuaciones de movimiento son

$$M\ddot{q} + C\dot{q} + Kq = b_2 u(t)$$

donde C es la matriz de amortiguamiento

$$solving\ \ for\ \ddot{q}\ and\ adding \quad \dot{q} = \dot{q}$$

$$\begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} 0 \\ M^{-1}b_2 \end{bmatrix} u(t)$$

$$\begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} 0 \\ M^{-1}b_2 \end{bmatrix} u(t)$$

$$x = \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix}$$

$$\dot{x} = A_c x + B_c u$$

↑ state vector

↑ state space equations

Mediciones

$$y(t) = C_c x(t) + D_c u(t)$$

En Resumen

$$\dot{x} = A_c x + B_c u$$

$$y(t) = C_c x(t) + D_c u(t)$$

$$A_c = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$ con ⇒ $$x = \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix}$$

(El objectivo es estimar {$A_{c,}$ $B_{c,}$ $C_{c,}$ D_c } a partir de observaciones de u(t) y y(t))

Es Única la Realización ? - NO

$$\dot{x}(t) = A_c x(t) + B_c u(t)$$

$$y(t) = C_c x(t) + D_c u(t)$$

$$x(t) = T \cdot z(t)$$

$$\dot{z}(t) = (T^{-1} A_c T) z(t) + (T^{-1} B_c) u(t)$$

$$y(t) = (C_c T) z(t) + D_c u(t)$$

Hay un numero infinito de triples {A,B,C} que llevan al mismo mapa

$$\dot{z}(t) = (T^{-1}A_c T)z(t) + (T^{-1}B_c)u(t)$$

$$y(t) = (C_c T)z(t) + D_c u(t)$$

$$x(t) = T \cdot z(t)$$

Si todo lo que podemos ver es entradas y salidas lo que se tiene es

$$\Rightarrow$$

$$T^{-1}A_c T$$

$$T^{-1}B_c$$

$$C_c T$$

$$D_c$$

Para un T indeterminado

$$T^{-1}A_cT$$

Todas las A, son “similares” lo que quiere decir que tiene los mismos autovalores.

LA SOLUCION

$$\dot{x} = A_c x + B_c u$$

$$x(t) = e^{A_c t} x(0) + \int_0^t e^{A_c (t-\tau)} B_c u(\tau)\, d\tau$$

Caracteristicas Modales

$$\dot{x} = A_c x + B_c u$$

$$A_c \Phi = \Phi \Lambda$$

$$A_c = \Phi \Lambda \Phi^{-1}$$

$$\dot{x} = \Phi \Lambda \Phi^{-1} x + B_c u$$

$$\Phi^{-1} \dot{x} = \Lambda \Phi^{-1} x + \Phi^{-1} B_c u$$

$$Y = \Phi^{-1} x$$

$$\dot{Y} = \Lambda Y + \Gamma u$$

$$y = C_c x = C_c \Phi Y$$

$$x(t) = e^{A_c t} x(0) + \int_0^t e^{A_c (t-\tau)} B_c u(\tau)\, d\tau$$

$$Y(t) = e^{\Lambda t} Y(0) + \int_0^t e^{\Lambda (t-\tau)} \Gamma_c u(\tau)\, d\tau$$

Vibracion Libre

$$Y(t) = e^{\Lambda t} Y(0)$$

$$\Lambda = diag(\lambda_1, \lambda_2 \ldots \lambda_N)$$

$$Y(t) = e^{\Lambda t} Y(0)$$

$$\begin{Bmatrix} Y_1(t) \\ Y_2(t) \\ . \\ Y_N(t) \end{Bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & . & \\ & & & e^{\lambda_N t} \end{bmatrix} \begin{Bmatrix} Y_{0,1} \\ Y_{0,2} \\ . \\ Y_{0,N} \end{Bmatrix}$$

$$Y_j(t) = e^{\lambda_j t} Y_{0,j} \qquad \lambda_j = \Re(\lambda_j) + \Im(\lambda_j) i = p_j + q_j i$$

$$Y_j(t) = e^{p_j t} . e^{i.q_j t} Y_{0,j}$$

$$Y_j(t) = Y_{0,j} e^{p_j t} (\cos(q_j t) + i \sin(q_j t))$$

$$Y_j(t) = Y_{0,j} e^{p_j t} (\cos(q_j t) + i \sin(q_j t))$$

$$y = \Psi Y$$

$$y(t) = \sum_{j=1}^{N} \psi_j Y_{0,j} e^{p_j t} (\cos(q_j t) + i \sin(q_j t))$$

Dado que la respuesta es real las contribuciones imaginarias se cancelan.

$$\lambda_j = \Re(\lambda_j) + \Im(\lambda_j) i = p_j + q_j i$$

$$p_j = -\omega_j \xi_j$$

$$q_j = \omega_{d,j} = \omega_j \sqrt{1 - \xi_j^{\,2}}$$

Formulacion en Tiempo Discreto

$$\dot{x} = A_c x + B_c u$$

$$x(t) = e^{A_c t} x(0) + \int_0^t e^{A_c (t-\tau)} B_c u(\tau)\, d\tau$$

$$x(t+\Delta t) = e^{A_c \Delta t} x(t) + e^{A_c \Delta t} \int_o^{\Delta t} e^{-A_c \tau'} B_c u(\tau')\, d\tau'$$

$$x(t+\Delta t) = e^{A_c \Delta t} x(t) + e^{A_c \Delta t} \int_o^{\Delta t} e^{-A_c \tau'} B_c u(\tau')\, d\tau'$$

consider the parameterization

$$u(\tau') = f_0(\tau') u(k) + f_1(\tau') u(k+1)$$

$$x(t+\Delta t) = e^{A_c \Delta t} x(t) + \left(e^{A_c \Delta t} \int_o^{\Delta t} e^{-A_c \tau'} B_c f_0(\tau')\, d\tau' \right) u(k) + \left(e^{A_c \Delta t} \int_o^{\Delta t} e^{-A_c \tau'} B_c f_1(\tau')\, d\tau' \right) u(k+1)$$

$$x(t+\Delta t) = \underbrace{e^{A_c \Delta t}}_{A_d} x(t) + \underbrace{\left(e^{A_c \Delta t} \int_o^{\Delta t} e^{-A_c \tau'} B_c f_0(\tau')\, d\tau' \right)}_{B_0} u(k) + \underbrace{\left(e^{A_c \Delta t} \int_o^{\Delta t} e^{-A_c \tau'} B_c f_1(\tau')\, d\tau' \right)}_{B_1} u(k+1)$$

$$x(k+1) = A_d x(k) + B_0 u(k) + B_1 u(k+1)$$

Considerece la ecuacion de salida

$$y(t) = C_c x(t) + D_c u(t)$$

En este caso se tiene simplemente

$$y(k) = C_c x(k) + D_c u(k)$$

$$x(k+1)=A_d x(k)+B_0 u(k)+B_1 u(k+1)$$

$$y(k)=C_c x(k)+D_c u(k)$$

Los algoritmos usuales no contienen el termino u(k+1)

$$x(k)=z(k)+B_1 u(k)$$

$$z(k+1)+B_1 u(k+1)=A_d\,(z(k)+B_1 u(k))+B_0 u(k)+B_1 u(k+1)$$

$$z(k+1)=A_d z(k)+(A_d B_1+B_0)u(k)$$

$$y(k)=C_c z(k)+(C_c B_1+D_c)u(k)$$

En Resumen:

$$z(k+1) = A_d z(k) + B_d u(k)$$

$$y(k) = C_d z(k) + D_d u(k)$$

$$\left.\begin{aligned} A_d &= e^{A_c \Delta t} \\ B_d &= A_d B_1 + B_0 \\ C_d &= C_c \\ D_d &= C_c B_1 + D_c \end{aligned}\right\}$$

$$B_j = \left(A_d \int_o^{\Delta t} e^{-A_c \tau'} B_c f_j(\tau')\, d\tau' \right)$$

Resultados para B_d y D_d dependen de la hipotesis sobre el comportamiento de la entrada en el intervalo DT

$$z(0) = x(0) - B_1 u(0)$$

D2C

$$A_d = e^{A_c \Delta t}$$

$$A_c = \frac{\log(A_d)}{\Delta t}$$

$$A_d = \Psi \Lambda \Psi^{-1}$$

$$\log(A_d) = \Psi \log(\Lambda) \Psi^{-1} \qquad \Lambda = diag(\lambda_1, \lambda_2 \ldots \lambda_N)$$

$$\lambda = re^{(\theta + 2\pi\ell)i}$$

$$\log(\lambda) = \log(r) + (\theta + 2\pi\ell)i$$

$$A_c = \Psi \begin{bmatrix} \cdot & & 0 \\ 0 & \frac{\log(r)}{\Delta t} + \left(\frac{\theta}{\Delta t} + \frac{2\pi\ell}{\Delta t}\right)i & \\ & & \cdot \end{bmatrix} \Psi^{-1}$$

$$A_c = \Psi \begin{bmatrix} \cdot & & 0 & \\ 0 & \frac{\log(r)}{\Delta t} + \left(\frac{\theta}{\Delta t} + \frac{2\pi\ell}{\Delta t} \right) i & \\ & & & \cdot \end{bmatrix} \Psi^{-1}$$

Optamos por ℓ = 0. Dado que los autovalores aparecen en pares complejos

$$-\pi \leq \theta \leq \pi$$

Por lo tanto el valor absoluto de la parte imaginaria del autovalor no puede ser mayor que

$$\frac{\pi}{\Delta t}$$

Este es el limite de Nyquist.

Como la parte imaginaria del autovalor es la frecuencia amortiguada

$$\frac{\pi}{\Delta t} \geq \omega_d$$

Si el paso de discretizacion no cumple con el limite de Nyquist las frecuencias identificadas son erróneas (aparecen “Aliased”).

RESUMEN

$$M\ddot{q} + C\dot{q} + Kq = b_2 u(t)$$

$$\dot{x} = A_c x + B_c u$$

$$y(t) = C_c x(t) + D_c u(t)$$

Autovalores de A_c son complejos – la parte real es el producto de la frecuencia y el amortiguamiento y la imaginaria es la frecuencia amortiguada

$$A_c = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \text{ provided } x = \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix}$$

C por los autovectores de A_c son los modos.

$$\dot{z}(t) = (T^{-1}A_c T)z(t) + (T^{-1}B_c)u(t)$$

$$y(t) = (C_c T)z(t) + D_c u(t)$$

En general

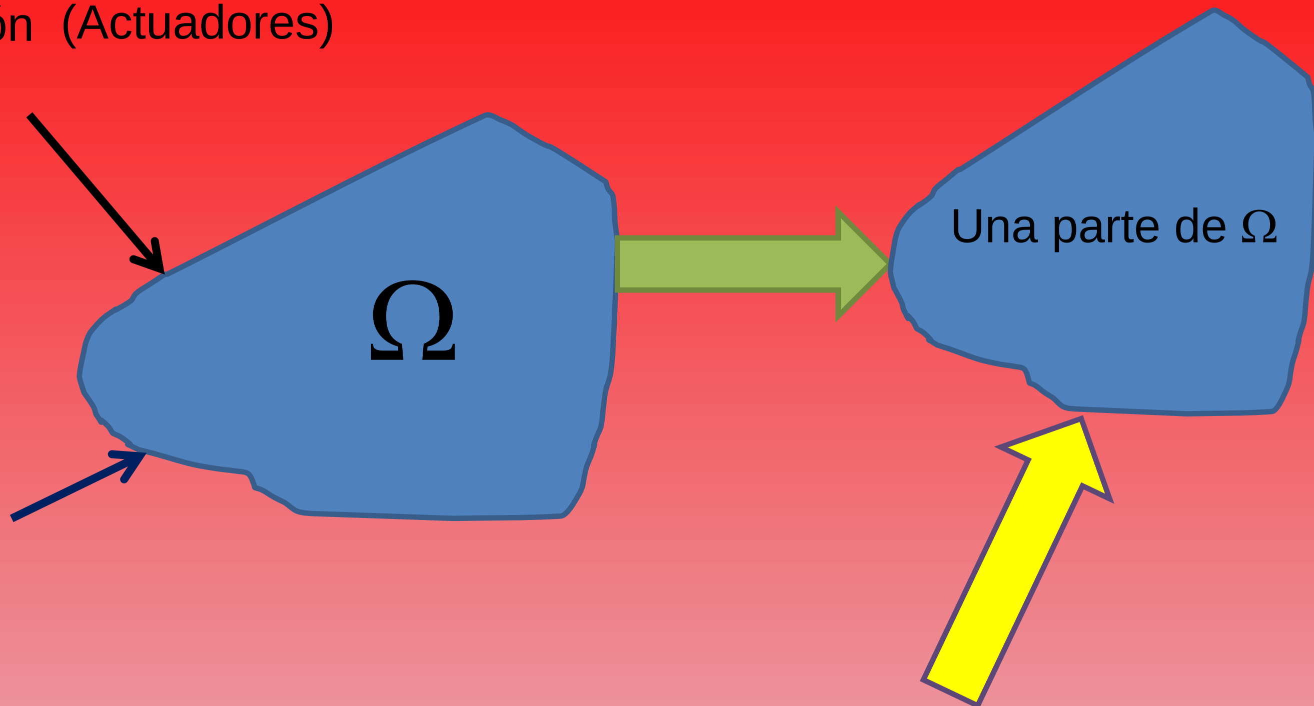
$$z(k+1) = A_d z(k) + B_d u(k)$$

$$y(k) = C_d z(k) + D_d u(k)$$

En tiempo discreto

CONTROLABILIDAD

Excitación (Actuadores)



Miramos (Sensores)

OBSERVABILIDAD

VEMOS UNA PARTE DE UNA PARTE DE Ω

Controlabilidad

El estado x_1 es controlable si existe una forma de lacanzarlo en tiempo finito dada la posicion de las entradas.

$$x(k+1) = A_d x(k) + B_d u(k)$$

Following the recurrence

$$x(1) = A_d x(0) + B_d u(0)$$

$$x(2) = A_d x(1) + B_d u(1) = A_d (A_d x(0) + B_d u(0)) + B_d u(1)$$

$$= A_d^2 x(0) + A_d B_d u(0) + B_d u(1)$$

$$x(k) = A_d^k x(0) + \sum_{j=0}^{k-1} A_d^{k-j-1} B_d u(j) \quad k \geq 0$$

$$x(k) = A_d^k x(0) + \sum_{j=0}^{k-1} A_d^{k-j-1} B_d u(j) \quad k \geq 0$$

$$x(k) - A_d^k x(0) = C_k U_k$$

$$C_k = \begin{bmatrix} B_d & A_d B_d & . & A_d^{k-1} B_d \end{bmatrix} \qquad U_k = \begin{Bmatrix} u(k-1) \\ u(k-2) \\ . \\ u(0) \end{Bmatrix}$$

La matriz C_k debe ser "full column rank".

Gramian de Controlabilidad

$$C_N = \begin{bmatrix} B_d & A_d B_d & . & A_d^{N-1} B_d \end{bmatrix}$$

Uncontrollable modes (if any) cannot be excited from the input locations and cannot, therefore be identified.

A system may be controllable but some modes may only be “marginally so” and thus may be difficult to identify. A practical question is, how to locate excitation sources to minimize controllability difficulties in a certain bandwidth. The controllability gramian offers guidance.

From the definition of controllability one finds that the input sequence that takes the system from the origin to the state x in k steps which has minimum norm is:

$$U_k = C_k^{-*} x$$

$$Cont\_gramian = C_k C_k^T$$

$$U(k) = C_k^T W_c^{-1} x$$

$$\left| U(k) \right|^2 = x^T W_c^{-1} x$$

The norm of the input signal needed to reach state x in k steps depends on the inverse of the controllability gramian – if the product is large it takes a lot of energy so this state is likely to be poorly excited.

$$|U(k)|^2 = x^T W_c^{-1} x$$

A possible approach for input location is then:

a) Assign all the possible locations to the set η = {n1, n2,… nq}

b) Using a model of the system compute, for each arrangement the controllability gramian

c) Normalize the modes of the discrete time system matrix to unity and obtain the energies.

d) Select the arrangement that minimizes the maximum energy.

Observabilidad

Se tiene

$$y(k) = C_d A_d^k x(0) + \sum_{j=0}^{k-1} C_d A_d^{k-j-1} B_d u(j) + D_d u(k)$$

El estado x(0) es observable si puede inferirse a partir de observaciones de entradas y salidas.

$$C_d A_d^k x(0) = y(k) - \sum_{j=0}^{k-1} C_d A_d^{k-j-1} B_d u(j) + D_d u(k)$$

$$\begin{bmatrix} C_d \\ C_d A_d \\ . \\ C_d A_d^{n-1} \end{bmatrix} x(0) = \Upsilon_n$$

Writing the equality for time stations k=0,1,2, …. n

So x(0) is computable without ambiguity if the matrix on the left side is full column rank. States on the null space of the observability matrix are unobservable (i.e., if the system is given an initial condition in the unobservable space and no loads act the measurements would remain identically zero). Cayley-Hamilton ensures that the rank of the observability does not increase after the power of the system matrix is N-1.

Gramian de Observabilidad

Observability matrix → $O_b = \begin{bmatrix} C_d \\ C_d A_d \\ . \\ C_d A_d^{N-1} \end{bmatrix}$

Since states that are not observable “cannot be seen” from the sensors, only observable states can be identified.

As in the case of controllability, an observable system may have some states that are marginally observable and it is of interest to have a quantitative way to measure observability. As one anticipates, the solution is offered by the observability gramian.

Small z means poor observability.

$$z = O_b x$$

$$|z|^2 = x^T O_b^T O_b x$$

Observability Provides Guidance for Locating Output Sensors

Observability in continuous time:

$$O_b(0,t_m) = \int_0^{t_m} e^{A_c^T \tau} C_c^T C_c e^{A_c \tau} d\tau$$

Which can be obtained as the solution to Lyapunov's equation

$$A_c^T O_b + O_b A_c + C_c^T C_c - e^{A_c^T t_m} C_c^T C_c e^{A_c t_m} = 0$$

Possible approach for output sensor locations:

a) Assign all the possible locations to the set g = {g1, g2,… gq}
b) Using a model of the system compute, for each arrangement the observability gramian
c) Normalize the modes of the discrete time system matrix to unity and obtain the norms $|z|^2 = x^T O_b^T O_b x$
d) Select the arrangement that maximizes the minimum norm.

IDENTIFICACION

(una versión)

Time-Domain Identification

$$y(k) = C_d A_d^k x(0) + \sum_{j=0}^{k-1} C_d A_d^{k-j-1} B_d u(j) + D_d u(k)$$

$$\ell = k - j$$

$$y(k) = C_d A_d^k x(0) + \sum_{\ell=1}^{k} C_d A_d^{\ell-1} B_d u(k-\ell) + D_d u(k)$$

$$y(k) = C_d A_d^k x(0) + \sum_{\ell=1}^{k} Y_\ell u(k-\ell) + D_d u(k)$$

$$Y_\ell = C_d A_d^{\ell-1} B_d$$

$\ell \geq 1$ Discrete-time Markov Parameters

$$y(k) = C_d A_d^k x(0) + \sum_{\ell=1}^{k} Y_\ell u(k-\ell) + D_d u(k)$$

taking $Y_0 = D_d$

$$y(k) = C_d A_d^k x(0) + \sum_{\ell=0}^{k} Y_\ell u(k-\ell)$$

The Markov Parameters are Basis Independent:

$$Y_\ell = C_d A_d^{\ell-1} B_d$$

$$Y_\ell = (C\,T)(T^{-1} A^{\ell-1} T)(T^{-1} B)$$

Remember that the zero MP is the direct transmission and this matrix is also basis independent.

$$y(k) = \sum_{\ell=0}^{p+1} Y_\ell u(k-\ell)$$

$$\begin{bmatrix} y_h & y_{h+1} & . & . & y_z \end{bmatrix} = \begin{bmatrix} Y_0 & Y_1 & . & . & Y_{p+1} \end{bmatrix} \begin{bmatrix} u(h) & u(h+1) & u(h+2) & . & u(z) \\ u(h-1) & u(h) & u(h+1) & . & . \\ . & . & u(h) & . & . \\ . & . & . & . & . \\ u(h-(p+1)) & u(h-p) & u(h-p+1) & . & u(z-(p+1)) \end{bmatrix}$$

$$y_{hz} = \Upsilon_{p+1} U_{hzp}$$

$$y_{hz} \in R^{mx(z-h+1)}$$

$$\Upsilon_{p+1} \in R^{mx(p+2)r}$$

$$U_{hzp} \in R^{(p+2)rx(z-h+1)}$$

Block Toeplitz

Doesn't work because the matrices are too big

The Observer Structure:

$$x(k+1) = A_d x(k) + B_d u(k)$$

$$y(k) = C_d x(k) + D_d u(k)$$

$$x(k+1) = A_d x(k) + B_d u(k) + Gy(k) - G(C_d x(k) + D_d u(k))$$

$$x(k+1) = (A_d - GC_d)x(k) + [(B_d - GD_d) \;\; G]\begin{Bmatrix} u(k) \\ y(k) \end{Bmatrix}$$

$$x(k+1) = (A_d - GC_d)x(k) + [(B_d - GD_d) \;\; G]\begin{Bmatrix} u(k) \\ y(k) \end{Bmatrix}$$

$$x(k+1) = \bar{A}x(k) + \bar{B}v(k)$$

Exactly the same form as the original expression except that the input now includes the true input and the output and the matrices are modified.

The output equation has not changed.

$$x(k+1) = \bar{A}x(k) + \bar{B}v(k)$$

$$x(1) = \bar{A}x(0) + \bar{B}v(0)$$

$$x(2) = \bar{A}x(1) + \bar{B}v(1) = \bar{A}(\bar{A}x(0) + \bar{B}v(0)) + \bar{B}v(1)$$

$$= \bar{A}^2 x(0) + \bar{A}\bar{B}v(0) + \bar{B}v(1)$$

$$x(k) = \bar{A}x(0) + \sum_{j=0}^{k-1} \bar{A}^{k-j-1}\bar{B}v(j) \quad k \geq 0$$

Substituting in the output equation

$$y(k) = C_d \bar{A}^k x(0) + \sum_{j=0}^{k-1} C_d \bar{A}^{k-j-1} \bar{B}\, v(j) + D_d u(k)$$

$$y(k) = \sum_{\ell=1}^{p+1} \bar{Y}_\ell v(k-\ell) + Y_0 u(k) \quad k \geq p+1$$

$$\bar{Y}_\ell = C_d \bar{A}^{\ell-1} \bar{B}$$

from this point it is a simple matter to get to the explicit linear system which now reads

$$y_{hz} = \bar{\Upsilon}_{p+1} V_{hzp}$$

$$x(k+1) = \bar{A}x(k) + \bar{B}v(k)$$

$$\bar{A} = (A_d - GC_d)$$

$$\bar{B} = [(B_d - GD_d) \;\; G]$$

$$\bar{Y}_\ell = C_d \bar{A}^{\ell-1} \bar{B}$$

$$\bar{Y}_0 = Y_0 = D_d$$

$$y_{hz} = \bar{\Upsilon}_{p+1} V_{hzp}$$

$$V_{hzp} = \begin{bmatrix} u(h) & u(h+1) & u(h+2) & . & u(z) \\ v(h-1) & v(h) & v(h+1) & . & . \\ . & . & v(h) & . & . \\ . & . & . & . & . \\ v(h-(p+1)) & v(h-p) & v(h-p+1) & . & v(z-(p+1)) \end{bmatrix}$$

$$\bar{\Upsilon}_{p+1} = \begin{bmatrix} \bar{Y}_0 & \bar{Y}_1 & . & . & \bar{Y}_{p+1} \end{bmatrix}$$

Extracting MP from OMP:

Markov Parameters

$$Y_j = C_d A^{j-1} B_d$$

$$Y_1 = C_d B_d$$

Observer Markov Parameters

$$\bar{Y}_j = C_d \bar{A}^{j-1} \bar{B}$$

$$\bar{Y}_j = C_d (A_d - GC_d)^{j-1} [(B_d - GD_d) \;\; G]$$

$$\bar{Y}_j^{(1)} = C_d (A_d - GC_d)^{j-1} (B_d - GD_d)$$

$$\bar{Y}_j^{(2)} = C_d (A_d - GC_d)^{j-1} G$$

for j =1

$$\bar{Y}_1^{(1)} = C_d B_d - C_d G D_d$$

$$Y_1 = \bar{Y}_1^{(1)} + \bar{Y}_1^{(2)} Y_0$$

$$\bar{Y}_1^{(2)} = C_d G$$

$$\bar{Y}_1^{(1)} = Y_1 - \bar{Y}_1^{(2)} Y_0$$

for j =2

$$\bar{Y}_2^{(1)} = C_d(A_d - GC_d)(B_d - GD_d)$$

$$\bar{Y}_2^{(1)} = C_dA_dB_d - C_dGC_dB_d - C_dA_dGD_d + C_dGC_dGD_d$$

$$C_dA_dB_d = \bar{Y}_2^{(1)} + C_dGC_dB_d + C_dA_dGD_d - C_dGC_dGD_d$$

$$\bar{Y}_2^{(2)} = C_d(A_d - GC_d)G$$

$$\bar{Y}_2^{(2)} = C_dA_dG - C_dGC_dG$$

$$C_dA_dG = \bar{Y}_2^{(2)} + C_dGC_dG$$

$$C_dA_dG = \bar{Y}_2^{(2)} + \bar{Y}_1^{(2)}\bar{Y}_1^{(2)}$$

$$Y_2 = \bar{Y}_2^{(1)} + \bar{Y}_1^{(2)}Y_1 + (\bar{Y}_2^{(2)} + \bar{Y}_1^{(2)}\bar{Y}_1^{(2)})Y_0 - \bar{Y}_1^{(2)}\bar{Y}_1^{(2)}Y_0$$

$$Y_2 = \bar{Y}_2^{(1)} + \bar{Y}_1^{(2)}Y_1 + \bar{Y}_2^{(2)}Y_0$$

for j =3

$$C_d A_d G = \bar{Y}_2^{(2)} + \bar{Y}_1^{(2)}\bar{Y}_1^{(2)}$$

$$\bar{Y}_3^{(1)} = C_d (A_d - GC_d)^2 (B_d - GD_d)$$

$$\bar{Y}_3^{(1)} = C_d (A_d^2 - A_d GC_d - GC_d A_d + GC_d GC_d)(B_d - GD_d)$$

$$\bar{Y}_3^{(1)} = C_d A_d^2 B_d - C_d A_d GC_d B_d - C_d GC_d A_d B_d + C_d GC_d GC_d B_d$$
$$-(C_d A_d^2 GD_d - C_d A_d GC_d GD_d - C_d GC_d A_d GD_d + C_d GC_d GC_d GD_d)$$

$$\bar{Y}_3^{(2)} = C_d A_d^2 G - C_d A_d GC_d G - C_d GC_d A_d G + C_d GC_d GC_d G$$

$$\bar{Y}_3^{(1)} = Y_3 - C_d A_d GC_d B_d - C_d GC_d A_d B_d + C_d GC_d GC_d B_d - \bar{Y}_3^{(2)} Y_0$$
$$Y_3 = \bar{Y}_3^{(1)} + (\bar{Y}_2^{(2)} + \bar{Y}_1^{(2)}\bar{Y}_1^{(2)})Y_1 + \bar{Y}_1^{(2)} Y_2 - \bar{Y}_1^{(2)}\bar{Y}_1^{(2)} Y_1 + \bar{Y}_3^{(2)} Y_0$$
$$Y_3 = \bar{Y}_3^{(1)} + \bar{Y}_2^{(2)} Y_1 + \bar{Y}_1^{(2)} Y_2 + \bar{Y}_3^{(2)} Y_0$$

$$Y_1 = \overline{Y}_1^{(1)} + \overline{Y}_1^{(2)} Y_0$$

$$Y_2 = \overline{Y}_2^{(1)} + \overline{Y}_1^{(2)} Y_1 + \overline{Y}_2^{(2)} Y_0$$

$$Y_3 = \overline{Y}_3^{(1)} + \overline{Y}_1^{(2)} Y_2 + \overline{Y}_2^{(2)} Y_1 + \overline{Y}_3^{(2)} Y_0$$

$$Y_k = \overline{Y}_k^{(1)} + \sum_{j=1}^{k} \overline{Y}_j^{(2)} Y_{k-j}$$

Since the non-zero OMP are 0,1,2… p --- the sum never has more than p+1 non-zero terms

From previous lectures we know how to extract Markov parameters from the data. Let's now define the block Hankel matrix H_k as:

$$H_k = \begin{bmatrix} Y_{k+1} & Y_{k+2} & \cdots & Y_{k+\beta} \\ Y_{k+2} & Y_{k+3} & \cdots & Y_{k+\beta+1} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{k+\alpha} & . & \cdots & Y_{k+\beta+\alpha-1} \end{bmatrix}$$

Where the number of α and β blocks is, at this point arbitrary.

We now recall that the observability block of order α is given by

$$P_\alpha = \begin{bmatrix} C_d \\ C_d A_d \\ \vdots \\ C_d A_d^{\alpha-1} \end{bmatrix}$$

And the controllability block of order beta is

$$Q_\beta = [B_d \quad A_d B_d \quad A_d^2 B_d \quad \cdots \quad A_d^{\beta-1} B_d]$$

It follows, therefore, that

$$P_\alpha A_d^k Q_\beta = \begin{bmatrix} C_d A_d^k B_d & C_d A_d^{k+1} B_d & \cdots \\ C_d A_d^{k+1} B_d & C_d A_d^{k+2} B_d & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

or

$$H_k = P_\alpha A_d^k Q_\beta = \begin{bmatrix} Y_{k+1} & Y_{k+2} & \cdot & Y_{k+\beta} \\ Y_{k+2} & Y_{k+3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ Y_{k+\alpha} & \cdot & \cdot & Y_{k+\alpha+\beta-1} \end{bmatrix}$$

In words: the block hankel matrix H_k is equal to the product of the observability block of order α time the system matrix to the power k times the controllability block of order β.

We have

$$H_0 = P_\alpha Q_\beta$$

Performing a Singular Value Decomposition:

$$H_0 = R \Sigma S^T$$

Recalling from previous lectures that the rank of the observability and controllability blocks is no more than the order of the system and that the rank of a product is never larger than the smaller rank of the multipliers we conclude that the rank of H_0 is no larger than N - independently of the size of the blocks α and β - where N is the system order.

Examining the dimensions we conclude that $H_0 \in R^{(\alpha m)x(\beta r)}$

$$H_0 = R \Sigma S^T$$

accepting that we've selected α and β large enough so they don't control the rank we find that Σ has N non-zero singular values – **which tells us the order of the system**

To ensure that the rank of H_0 is not controlled by the number of blocks we select these block sizes such that

$$\alpha \geq \frac{N}{m} \qquad \beta \geq \frac{N}{r}$$

We have, therefore

$$H_0 = \begin{bmatrix} R_s & R_n \end{bmatrix} \begin{bmatrix} \Sigma_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_s & V_n \end{bmatrix}^T$$

$$H_0 = R_s \Sigma_s V_s^T$$

We can express the non-zero singular values as a product of two diagonal matrices

$$\Sigma_s = E_1 E_2$$

and write

$$H_0 = (R_s E_1)(E_2 V_s^T)$$

$$H_0 = (R_s E_1)(E_2 V_s^T)$$

and recalling that

$$H_0 = P_\alpha Q_\beta$$

It is apparent that we can set

$$P_\alpha = R_s E_1 \qquad\qquad Q_\beta = E_2 V_s^T$$

$$P_\alpha = R_s E_1$$

But we recall that

$$P_\alpha = \begin{bmatrix} C_d \\ C_d A_d \\ \vdots \\ C_d A_d^{\alpha-1} \end{bmatrix}$$

A realization of the state to output matrix C_d is given, therefore, by the first m rows of the matrix → $R_s E_1$

We also have

$$Q_\beta = E_2 V_s^T$$

where

$$Q_\beta = [B_d \quad A_d B_d \quad A_d^2 B_d \quad \cdots \quad A_d^{\beta-1} B_d]$$

A realization of the input to state matrix B_d is given, therefore, by the first r columns of the matrix

$$E_2 V_s^T$$

We now face the question of how to compute the system matrix A_{d}.

There are various ways to go about it, a commonly used one uses the block Hankel matrix H_1 as illustrated next.

$$H_1 = P_\alpha A_d Q_\beta = (R_s E_1) A_d (E_2 S_s^T)$$

Recognizing the orthogonality of the matrices R_s and S_s one gets

$$A_d = E_1^{-1} R_n^T H_1 S_n E_2^{-1}$$

And since D_d is the first Markov parameter we have obtained a full realization

ERA Summary:

1. Select values of α and β and form the block hankel matrices H_0 and H_1

2. Perform a SVD of H_0 and determine the system order (N) by inspecting the non-zero singular values.

3. Factor the non-zero singular values into a product – the typical approach is to use $E_1 = E_2$ which is known as balanced realization

4. Obtain a realization of C_d and B_d from the SVD decomposition of H_0

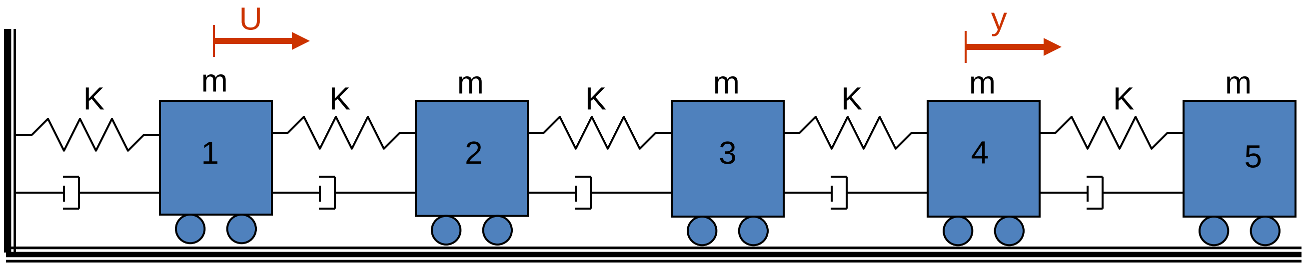
5. Obtain the discrete time system matrix using

$$A_d = E_1^{-1} R_n^T H_1 S_n E_2^{-1}$$

Only the CONTROLLABLE and OBSERVABLE part of the system can be identified from observations.

Controllability is determined by the number and position of the excitation sources (these may be known or unknown excitations).

Observability is determined by the number and position of output sensors



m=1, k=487.303 stiffness proportional damping.

Mode Number	Angular Frequency (w)	Damping Ratio
1	6.2832	0.02
2	18.3405	0.0584
3	28.9120	0.0920
4	37.1413	0.1182
5	42.3615	0.1348

$$M\ddot{q} + C_{dam}\dot{q} + Kq = b_2 U(t)$$

K=

974.61	-487.30	0.00	0.00	0.00
-487.30	974.61	-487.30	0.00	0.00
0.00	-487.30	974.61	-487.30	0.00
0.00	0.00	-487.30	974.61	-487.30
0.00	0.00	0.00	-487.30	487.30

M=

1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

C_{dam}=

6.20	-3.10	0.00	0.00	0.00
-3.10	6.20	-3.10	0.00	0.00
0.00	-3.10	6.20	-3.10	0.00
0.00	0.00	-3.10	6.20	-3.10
0.00	0.00	0.00	-3.10	3.10

b_2=

0
1
0
0
0

$$\dot{X} = AX + BU$$
$$y = CX + DU$$

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}K \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ M^{-1}b_2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 | 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \end{bmatrix}$$

Discrete to Continuous Conversion with Zero Order Hold (ZOH) assumption

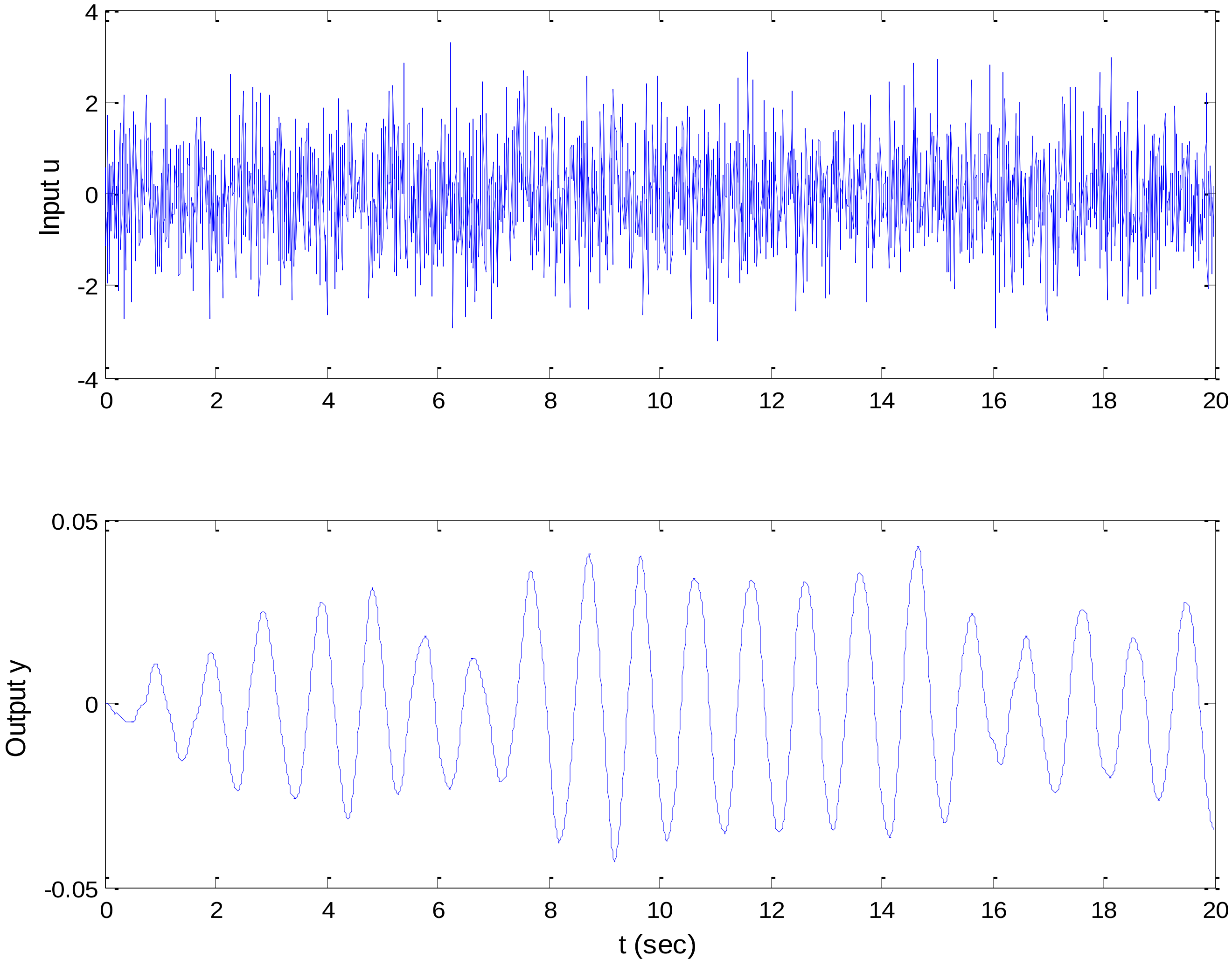
$$A_d = e^{A\Delta t}$$

$$B_d = (A_d - I)A^{-1}B$$

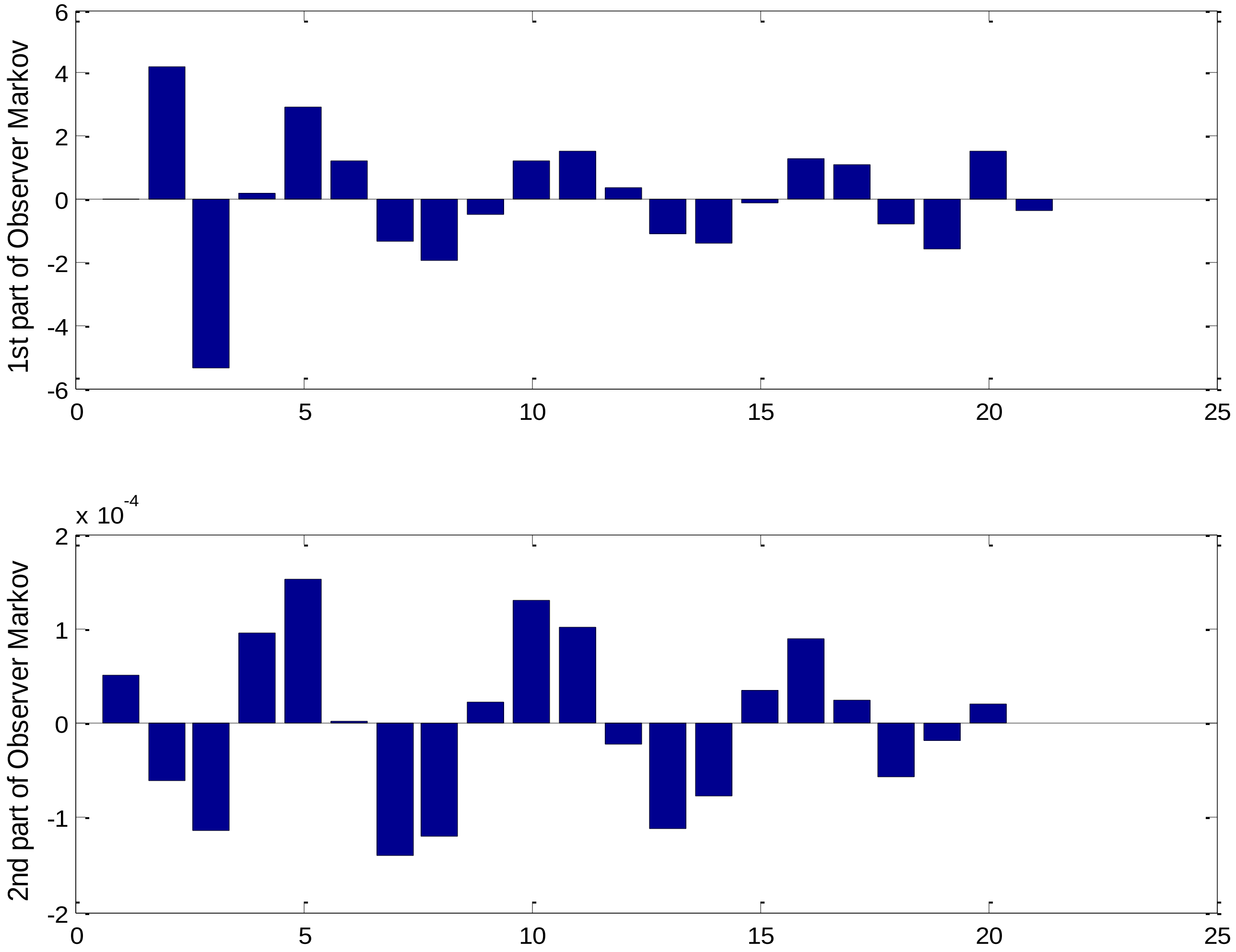
$$C_d = C$$

$$D_d = D$$

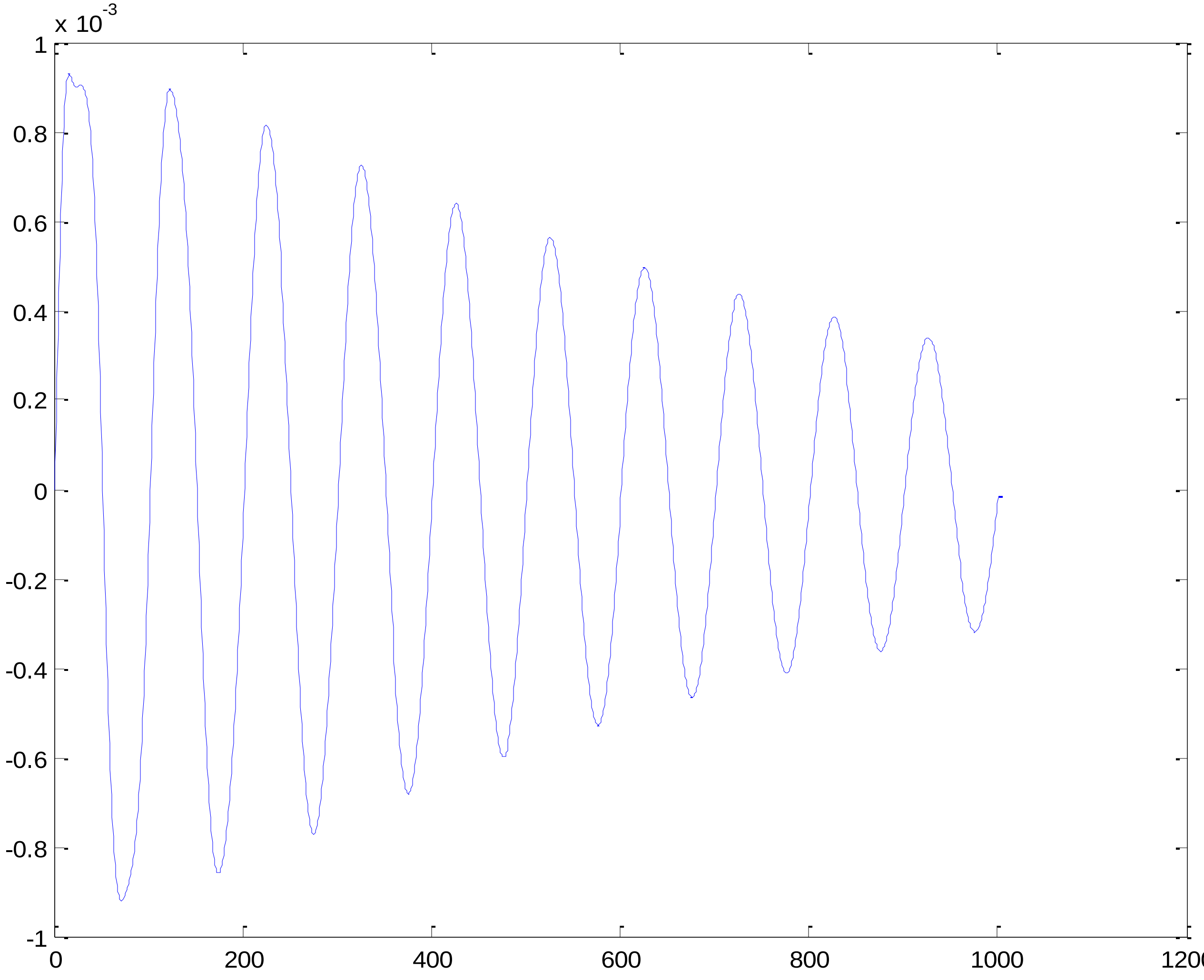
$$X_{k+1} = A_d X_k + B_d U_k$$
$$y_k = C_d X_k + D_d U_k$$


4
2
0
-2
-4
Input u
0
2
4
6
8
10
12
14
16
18
20
0.05
0
-0.05
Output y
0
2
4
6
8
10
12
14
16
18
20
t (sec)

Observer Markovs (assumed to be 20 non zeros)

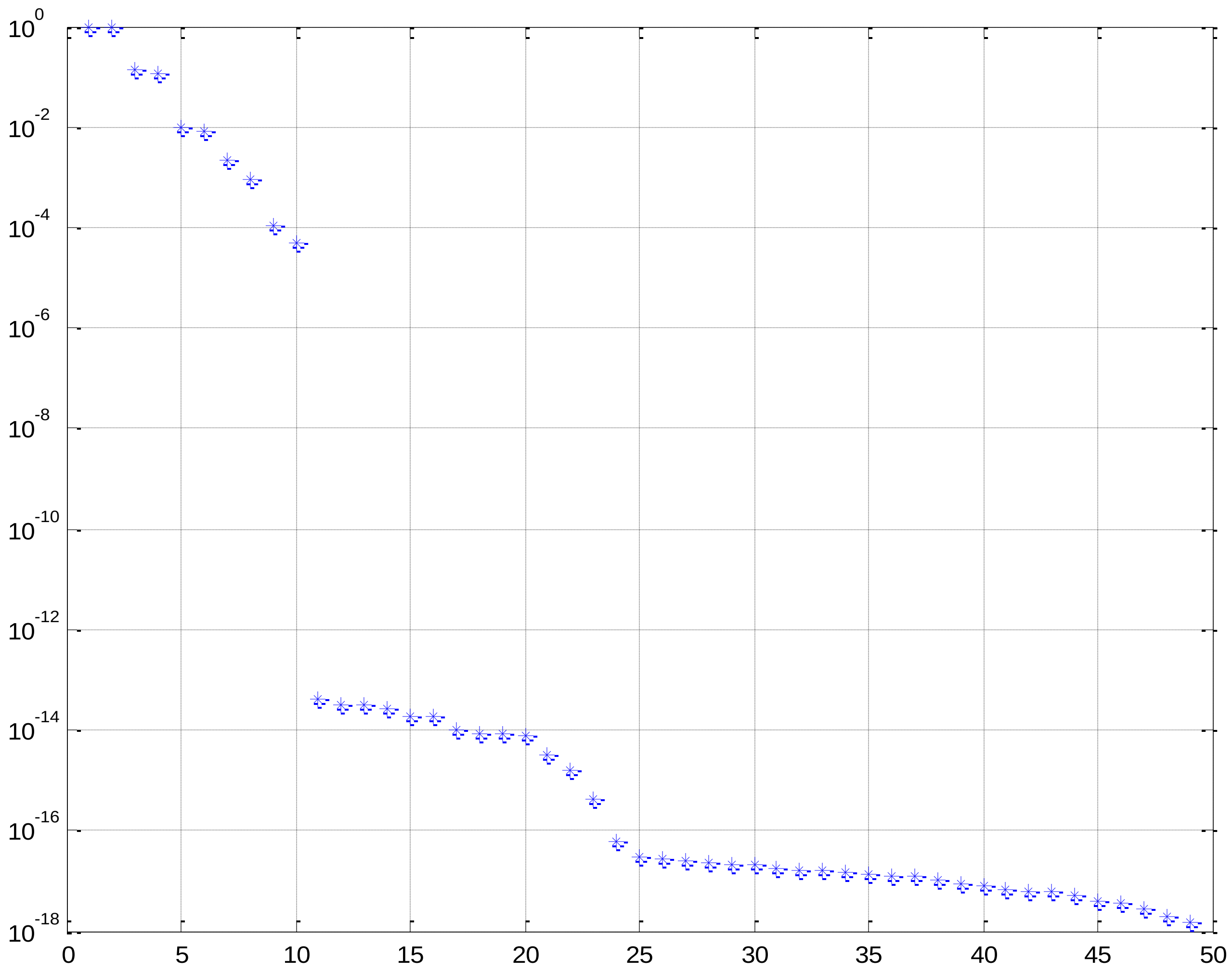


Markov Parameters



H_0 =1e-4*

0.50	1.50	2.49	3.47	4.42	5.32	6.16	6.91	7.57	8.12	8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03
1.50	2.49	3.47	4.42	5.32	6.16	6.91	7.57	8.12	8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00
2.49	3.47	4.42	5.32	6.16	6.91	7.57	8.12	8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99
3.47	4.42	5.32	6.16	6.91	7.57	8.12	8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99
4.42	5.32	6.16	6.91	7.57	8.12	8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00
5.32	6.16	6.91	7.57	8.12	8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01
6.16	6.91	7.57	8.12	8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03
6.91	7.57	8.12	8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03
7.57	8.12	8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02
8.12	8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02	9.00
8.56	8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02	9.00	8.97
8.89	9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02	9.00	8.97	8.91
9.11	9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02	9.00	8.97	8.91	8.84
9.24	9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02	9.00	8.97	8.91	8.84	8.73
9.29	9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02	9.00	8.97	8.91	8.84	8.73	8.61
9.29	9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02	9.00	8.97	8.91	8.84	8.73	8.61	8.45
9.25	9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02	9.00	8.97	8.91	8.84	8.73	8.61	8.45	8.25
9.20	9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02	9.00	8.97	8.91	8.84	8.73	8.61	8.45	8.25	8.02
9.13	9.07	9.03	9.00	8.99	8.99	9.00	9.01	9.03	9.03	9.02	9.00	8.97	8.91	8.84	8.73	8.61	8.45	8.25	8.02	7.73



10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
10^-16
10^-18
0
5
10
15
20
25
30
35
40
45
50

$$H_0 = U\Sigma V^t = (U\Sigma^{\frac{1}{2}})(\Sigma^{\frac{1}{2}} V^t) = PQ$$

P=

-0.027	0.020	0.018	0.005	0.004	-2.51E-03	1.59E-04	-3.67E-04	6.40E-06	-7.80E-06
-0.028	0.018	0.015	0.002	0.002	0.0000	0.0001	0.0002	0.0000	0.0000
-0.029	0.016	0.012	-0.001	0.000	0.0014	-0.0001	0.0003	0.0000	0.0000
-0.030	0.015	0.008	-0.003	-0.002	0.0018	-0.0002	0.0002	0.0000	0.0000
-0.031	0.013	0.005	-0.005	-0.003	0.0015	-0.0002	0.0000	0.0000	0.0000
-0.031	0.012	0.001	-0.006	-0.003	0.0008	-0.0001	-0.0002	0.0000	0.0000
-0.032	0.011	-0.002	-0.006	-0.003	-0.0001	0.0000	-0.0002	0.0000	0.0000
-0.032	0.009	-0.004	-0.006	-0.002	-0.0010	0.0001	-0.0002	0.0000	0.0000
-0.032	0.008	-0.006	-0.005	-0.001	-0.0016	0.0002	-0.0001	0.0000	0.0000
-0.032	0.007	-0.007	-0.004	0.000	-0.0019	0.0002	0.0000	0.0000	0.0000
-0.032	0.006	-0.008	-0.002	0.001	-0.0017	0.0001	0.0001	0.0000	0.0000
-0.032	0.005	-0.009	-0.001	0.002	-0.0012	0.0000	0.0002	0.0000	0.0000
-0.032	0.004	-0.008	0.000	0.003	-0.0005	-0.0001	0.0002	0.0000	0.0000
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.

Realization of C_d

Q=

-0.026	-0.027	-0.028	-0.029	-0.030	.	.	.
-0.019	-0.017	-0.015	-0.014	-0.012	.	.	.
-0.018	-0.015	-0.012	-0.009	-0.005	.	.	.
0.005	0.001	-0.002	-0.004	-0.005	.	.	.
0.004	0.002	0.001	-0.001	-0.002	.	.	.
0.002	0.000	-0.001	-0.002	-0.002	.	.	.
-7.0E-05	-1.2E-04	-3.9E-05	7.5E-05	1.5E-04	.	.	.
-4.0E-04	1.2E-04	2.9E-04	2.4E-04	8.6E-05	.	.	.
-5.9E-06	2.3E-06	6.6E-06	4.7E-06	-5.8E-07	.	.	.
-8.7E-06	1.2E-05	6.5E-06	-3.2E-06	-7.9E-06	.	.	.

Realization of B_d

$$A_d = \Sigma^{\frac{-1}{2}} U^t H_1 V \Sigma^{\frac{-1}{2}}$$

Realization of A_d

Converting the discrete realizations to continuous (ZOH)

$$A_c = \frac{\log m(A_d)}{\Delta t}$$

$$B_c = A(A_d - I)^{-1} B_d$$

$$C_c = C_d$$

$$D_c = D_d$$

Jordan Form

$$\dot{X} = A_c X + B_c U$$
$$y = C_c X + D_c U$$

$$A_c = \Phi \Lambda \Phi^{-1}$$
$$X = \Phi Y$$

$$\dot{Y} = \Lambda Y + \Phi^{-1} B_c U$$
$$y = C_c \Phi Y + D_c U$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \\ & & & \lambda_n \end{bmatrix}$$

$$\lambda_j = -w\zeta \pm w\sqrt{1-\zeta^2}\,\hat{i}$$

Mode Number	λ	Angular Frequency (w)	Damping Ratio ζ
1	-5.7121 + 41.975 i	6.2832	0.02
2	-4.391 + 36.881 i	18.3405	0.0584
3	-2.6608 + 28.789 i	28.9120	0.0920
4	-1.0707 + 18.309 i	37.1413	0.1182
5	-0.12566 + 6.2819 i	42.3615	0.1348

Recalling the true values

Mode Number	Angular Frequency (w)	Damping Ratio
1	6.2832	0.02
2	18.3405	0.0584
3	28.9120	0.0920
4	37.1413	0.1182
5	42.3615	0.1348

Introduction to Stochastic Identification

We assume that the output data is a realization of the system

$$x(k+1) = Ax(x) + w(k)$$

$$y(k) = Cx(k) + v(k)$$

where ω and v are assumed to be uncorrelated zero mean white noise processes of unknown co-variance.

It is assumed that the stochastic process is stationary, and the state is uncorrelated with both process noise and the output noise

$$E(x_k v_k^T) = 0$$

$$E(x_k w_k^T) = 0$$

$$E(x_{k+1} x_{k+1}^T) = \Sigma$$ ← not a function of k

$$E\ [(Ax_k + w_k)(Ax_k + w_k)^T] = \Sigma$$

$$E(Ax_k w_k^T) + E(Ax_k x_k^T A^T) + E(w_k w_k^T) + E(w_k x_k^T A^T) = \Sigma$$

$$E(Ax_k w_k^T) + E(Ax_k x_k^T A^T) + E(w_k w_k^T) + E(w_k x_k^T A^T) = \Sigma$$

which simplifies to:

$$\Sigma = A \Sigma A^T + Q$$

where

$$Q = E(w_k w_k^T)$$

We define the output covariance function matrix Λ_i for a shift *i* as

$$\Lambda_i = E(y_{k+i}\, y_k^T)$$

Substituting the output equation one gets

$$\Lambda_i = E[(Cx_{k+i} + v_{k+i})(Cx_k + v_k)^T]$$

$$\Lambda_i = E(C\,x_{k+i}v_k^T) + E(C\,x_{k+i}x_k^T C^T) + E(v_{k+i}v_k^T) + E(v_{k+i}x_k^T C^T)$$

Assuming that the measurement noise is white, i.e., $E(v_{k+i}v_k^T) = 0$

(except for *i*=0) and recalling that $E(x_k v_k^T) = 0$

$$\Lambda_i = E(Cx_{k+i}\,x_k^T C^T)$$

$$\Lambda_i = E(C x_{k+i} x_k^T C^T)$$

$$\Lambda_i = C E(x_{k+i} x_k^T) C^T$$

From the state recurrence it is a simple matter to show that

$$x_{k+i} = A^i x_k + A^{i-1} w_k + A^{i-2} w_{k+1} + \cdots + w_{k+i-1}$$

Post-multiplying x_k^T and taking expectations

$$E(x_{k+i} x_k^T) = A^i E(x_k x_k^T) = A^i \Sigma$$

$$\Lambda_i = C A^i \Sigma C^T$$

Looking back two slides one notes that for zero shift the result is

$$\Lambda_0 = C\,\Sigma\,C^T + R$$ i=0

and for i $\neq$ 0

$$\Lambda_i = C\,A^i\,\Sigma\,C^T$$

$$R = E(v_k v_k^T)$$

defining $$G = E(x_{k+1} y_k^T)$$

$$G = E[(Ax_k + \omega_k)(Cx_k + \nu_k)^T]$$

$$G = E[Ax_k \nu_k^T + Ax_k x_k^T C^T + \omega_k \nu_k^T + \omega_k x_k^T C^T]$$

$$\Downarrow$$

$$G = A\Sigma C^T$$

from last slide

$$\Lambda_i = C A^i \Sigma C^T$$

$$\Downarrow$$

$$\Lambda_i = C A^{i-1} A \Sigma C^T$$

$$\Lambda_i = C A^{i-1} G$$

$$\Lambda_i = C\,A^{i-1}G$$

Recalling that the Markov Parameters in the known input case are given by

$$Y_i = C\,A^{i-1}B$$

We conclude that the output covariance can be treated as Markov Parameters of a system having the matrices {A,G,C}

We can, therefore, estimate the output covariance and from there join our ERA algorithm at the point where the Hankel matrices are formed and proceed without any changes to obtain a realization for {A,G,C}

Once the realization is obtained one can get the covariance of the measurement noise from

$$G = A\Sigma C^T \qquad \Lambda_0 = C\,\Sigma\,C^T + R$$

$$A^{-1}G = \Sigma\,C^T \qquad \Lambda_0 = C\,A^{-1}G + R$$

$$R = \Lambda_0 - C\,A^{-1}G$$