# Control of Delay-ODE, Delay-PDE, and PDE-ODE Cascades

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## I. Introduction

Delay systems occupy a special place within the field of distributed parameter systems. Certain problems for this class of distributed parameter systems lend themselves to more elegant, or even explicit solutions.

An enormous wealth of knowledge and research results exists for control of systems with state delays and input delays. Problems with long input delays, for unstable plants, represent a particular challenge.

It has been known for about three decades now [1], [2] that the system

$$\dot{X}(t) = AX(t) + BU(t-D)\,, \tag{1}$$

where $D$ is an arbitrarily long delay and $(A,B)$ is a controllable pair, can be stabilized with the infinite-dimensional 'predictor' feedback

$$U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right], \tag{2}$$

where the gain $K$ is chosen so that the matrix $A+BK$ is Hurwitz.

The predictor feedback (2) represents a particular form of 'boundary control,' commonly encountered in the context of control of partial differential equations. Motivated by our recent advances in solving boundary control problems for various classes of PDEs [3], [4], we have considered various extensions to the problem of LTI ODE systems with input delays. These extensions are the subject of our upcoming book [5].

## II. Delay-ODE Cascades: Adaptive and Nonlinear

The key to various extensions that we develop in [5] is the observation that one can find a Lyapunov functional for the system (1), (2) as

$$V(t) \;=\; X(t)^T PX(t) + \frac{a}{2}\int_{t-D}^{t}(1+\theta+D-t)W(\theta)^2 d\theta\,, \tag{3}$$

where $P$ is the solution of the Lyapunov equation $P(A+BK)+(A+BK)^{\mathrm{T}}P = -Q$ and the function $W(\theta)$ is given by the invertible 'backstepping' (identity plus a Volterra operator) transformation

$$W(\theta) \;=\; U(\theta) - K\left[\int_{t-D}^{\theta}\mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma + \mathrm{e}^{A(\theta+D-t)}X(t)\right], \tag{4}$$

with $-D \le t-D \le \theta \le t$.

From this point on, the ability to construct a Lyapunov functional can be exploited in various ways, including deriving disturbance attenuation estimates when the system (1) is subject to an additive disturbance, proving robustness to a small (positive or negative) error in $D$, conducting an inverse optimal redesign of the predictor feedback law (which results in a feedback law which minimizes a cost functional with a positive definite penalty on $X(t)$, $U(t)$, and $\dot{U}(t)$), and solving the following two major problems:

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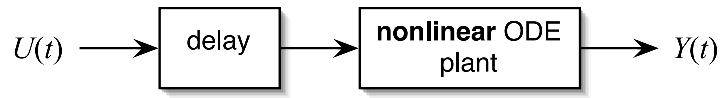
- adaptive control in the presence of a completely unknown and arbitrarily long $D$,
- predictor feedback design for some classes of nonlinear systems.

The delay-adaptive design uses a certainty-equivalence form of the feedback law (2) and employs a particular update law for the estimate $\hat{D}(t)$ of the unknown delay $D$. Starting from a severely overestimated $\hat{D}(0)$, or from an initial estimate as small as $\hat{D}(0) = 0$, the adaptive controller stabilizes an unstable LTI plant with an unknown $D$.




The nonlinear predictor design is developed for two classes of systems. For the broad class of *forward complete* systems, i.e., systems that do not exhibit a finite escape time for any initial condition and any input signals that remain finite over finite time intervals, which includes many mechanical and other systems, predictor feedback is developed which achieves global asymptotic stability, as long as the system without delay is globally asymptotically stabilizable. However, the 'predictror' requires the solution of a nonlinear integral equation (or a nonlinear DDE) in real time.

For the *strict-feedforward* subclass of forward complete systems, we develop an explicit formula for the predictor, and thus an explicit formula for the stabilizing feedback law in the presence of delay of any length. The following example illustrates this concept. The following third-order strict-feedforward system, which is not feedback linearizable,

$$\dot{Z}_1(t) = Z_2(t) + Z_3^2(t) \tag{5}$$

$$\dot{Z}_2(t) = Z_3(t) + Z_3(t)U(t-D) \tag{6}$$

$$\dot{Z}_3(t) = U(t-D)\,, \tag{7}$$

is globally asymptotically stabilized by the predictor feedback

$$\begin{aligned} U(t) = & -P_1(t) - 3P_2(t) - 3P_3(t) - \frac{3}{8}P_2^2(t) + \frac{3}{4}P_3(t)\left(-P_1(t) - 2P_2(t) + \frac{1}{2}P_3(t) + \frac{P_2(t)P_3(t)}{2}\right. \\ & \left. + \frac{5}{8}P_3^2(t) - \frac{1}{4}P_3^3(t) - \frac{3}{8}\left(P_2(t) - \frac{P_3^2(t)}{2}\right)^2\right), \end{aligned} \tag{8}$$

where the $D$-second-ahead predictor of $(X_1(t), X_2(t), X_3(t))$ is given explicitly by

$$\begin{aligned} P_1(t) = & Z_1(t) + DZ_2(t) + \frac{1}{2}D^2 Z_3(t) + DZ_3^2(t) + 3Z_3(t)\int_{t-D}^{t}(t-\theta)U(\theta)d\theta \\ & + \frac{1}{2}\int_{t-D}^{t}(t-\theta)^2 U(\theta)d\theta + \frac{3}{2}\int_{t-D}^{t}\left(\int_{t-D}^{\theta}U(\sigma)d\sigma\right)^2 d\theta \end{aligned} \tag{9}$$

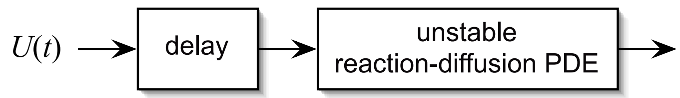
$$P_2(t) = Z_2(t) + DZ_3(t) + Z_3(t)\int_{t-D}^{t}U(\theta)d\theta + \int_{t-D}^{t}(t-\theta)U(\theta)d\theta + \frac{1}{2}\left(\int_{t-D}^{t}U(\theta)d\theta\right)^2 \tag{10}$$

$$P_3(t) = Z_3(t) + \int_{t-D}^{t}U(\theta)d\theta\,. \tag{11}$$

Note that the nonlinear infinite-dimensional feedback operator employs a finite Volterra series in $U(\theta)$.

## III. Delay-PDE Cascades

When a plant with an input delay is a PDE, this introduces special challenges, particularly if the PDE is actuated through boundary control, which makes the $B$ operator unbounded. In [5] we consider two benchmark delay-PDE cascades, one where the plant is a parabolic PDE and the other where the plant is a second-order hyperbolic PDE. We review here the parabolic case, where the plant is an unstable reaction-diffusion equation with an arbitrarily large number of unstable eigenvalues in open loop.




Consider the PDE system

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) \tag{12}$$
$$u(0,t) = 0 \tag{13}$$
$$u(1,t) = U(t-D), \tag{14}$$

where $\lambda$ is an arbitrary constant. We derive a stabilizing feedback law in the explicit form,

$$\begin{aligned} U(t) &= 2\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{I_1\left(\sqrt{\lambda\,(1-\xi^2)}\right)}{\sqrt{\lambda\,(1-\xi^2)}} d\xi \\ &\times \sum_{n=1}^{\infty}\left(-\mathrm{e}^{(\lambda-\pi^2 n^2)D}\int_0^1 \sin(\pi n y)u(y,t)dy + \pi n(-1)^n \int_{t-D}^{t} \mathrm{e}^{(\lambda-\pi^2 n^2)(t-\theta)}U(\theta)d\theta\right), \end{aligned} \tag{15}$$

where $I_1(\cdot)$ is a Bessell function.

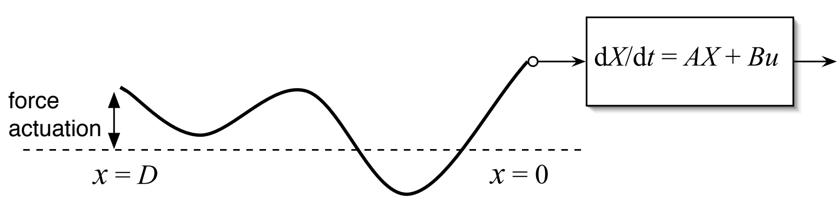
## IV. PDE-ODE Cascades

The third major class of problems that we consider in [5], which represents a seamless transition from the classical predictor feedback (2), are PDE-ODE cascades. This is motivated by the observation that (1) is a boundary control problem for a cascade of a transport PDE and an ODE.

In [5] we consider three classes of PDEs in cascade with ODEs:

- first-order hyperbolic PDEs (the pure delay is a special case in this class),
- parabolic PDEs,
- second-order hyperbolic PDEs.

We review the solution to the third problem in this list here.




Consider an ODE controlled through the undamped wave (string) PDE:

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{16}$$
$$u_{tt}(x,t) = u_{xx}(x,t) \tag{17}$$
$$u_x(0,t) = 0 \tag{18}$$
$$u(D,t) = U(t), \tag{19}$$

The explicit feedback law is designed as

$$U(t) = K\Sigma(D,c)X(t) + \int_0^D \varphi(D-y)u(y,t)dy + \int_0^D \psi(D-y)u_t(y,t)dy, \tag{20}$$

where

$$
\begin{aligned}
\Sigma(D,c) &= M(D)+c\int_0^D M(y)Ady & (21)\\
M(x) &= \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix} & (22)\\
\varphi(\tau) &= \mu(\tau)+cK(I+M(\tau))B & (23)\\
\mu(s) &= \int_0^s KM(\xi)ABd\xi & (24)\\
\psi(\tau) &= m(\tau)-c+c\int_0^\tau \mu(\eta)d\eta & (25)\\
m(s) &= \int_0^s KM(\xi)Bd\xi & (26)
\end{aligned}
$$

and $c > 0$.

## V. Other Problems and Tools

We consider a few other problems in [5], including

- observer problems for system with sensor delays,
- observer problems with sensor dynamics modeled by PDEs,
- predictor feedback for systems with time-varying delays.

Three major technical elements should be of interest and possible use for many researchers interested in delay systems:

1) the construction of backstepping transformations that allow one to deal with delays and PDE dynamics at the input (or in the main line of applying control action, such as in the chain of integrators for systems in triangular forms),
2) the construction of Lyapunov functionals and explicit stability estimates, with the help of direct and inverse backstepping transformations,
3) the connection between delay systems and an array of other classes of PDE systems which can be approached in a similar manner, each introducing a different type of challenges.

## References


[1] A. Z. Manitius and A. W. Olbrot, "Finite spectrum assignment for systems with delays," *IEEE Trans. on Automatic Control*, vol. 24, pp. 541–553, 1979.
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