Stability Analysis of Three-Agent Consensus Dynamics with Fixed Topology & Three Non-Identical Delays

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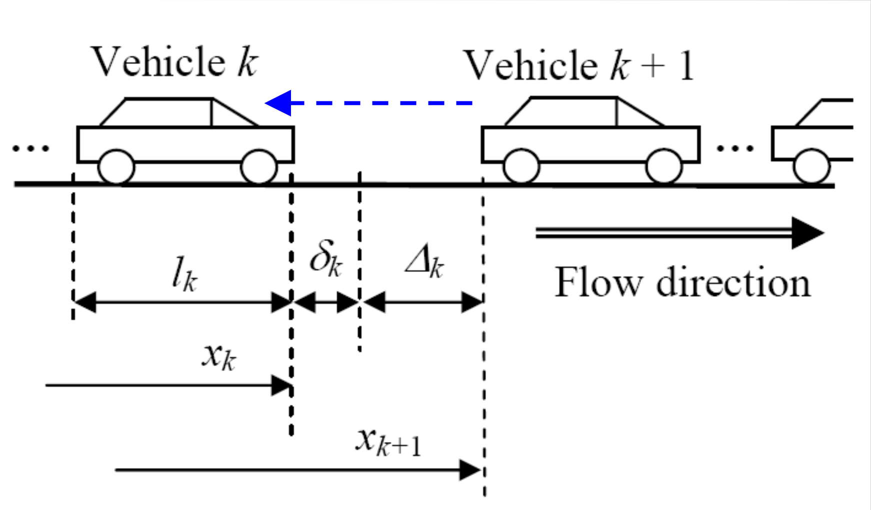
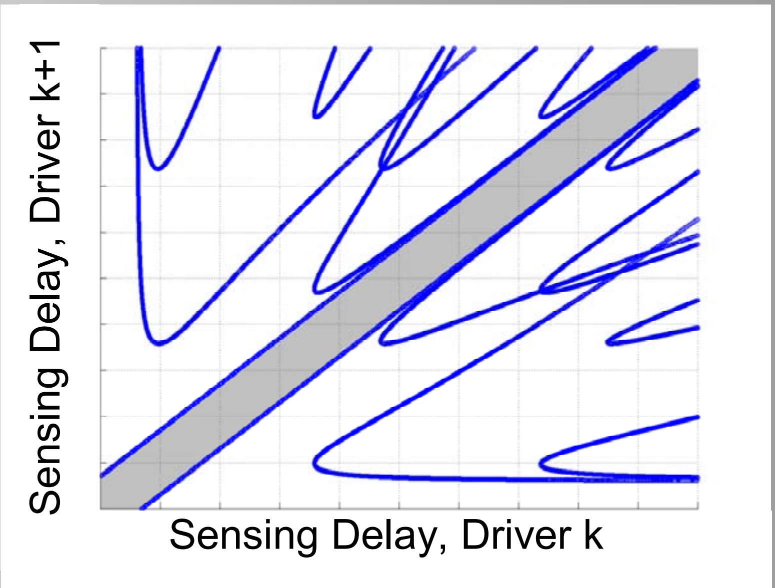
1D: Vehicular Traffic Flow*



Asymptotic Stability (shaded)



Velocity synchronization

$$\dot{v}_k(t) = \alpha_k(v_{k+1}(t-\tau_{k+1}) - v_k(t-\tau_k)),$$

Driver reaction delays

* Bose & Ioannau (2003), Sipahi & Niculescu (2006-2009)

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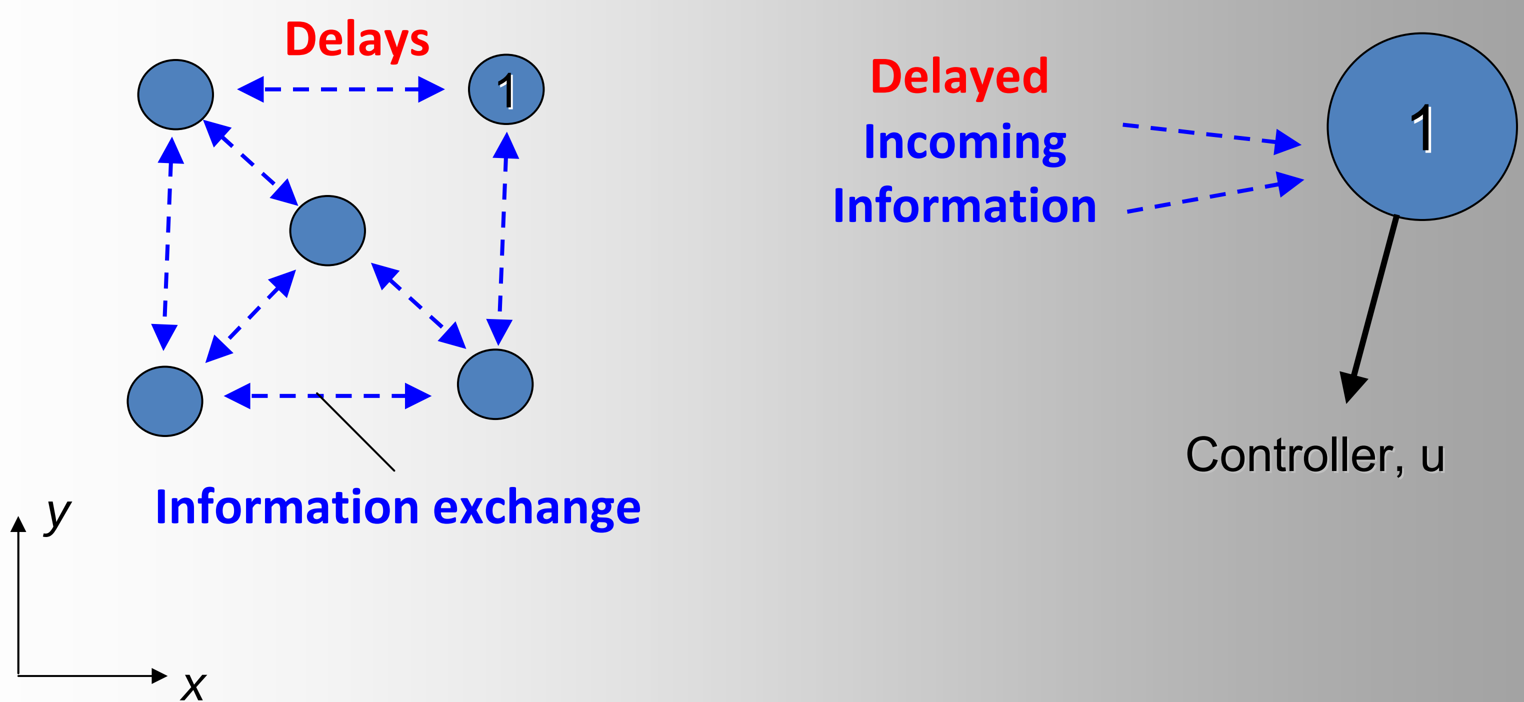
2D: Consensus of Agents*
(Top View)



Mathematical Model

Without delays

$$\frac{d}{dt}x_k(t) = -\sum_{\text{Topology}} \alpha_{kj}\left(x_k(t) - x_j(t)\right)$$

Delayed Information

*With delays**

$$\frac{d}{dt}x_k(t) = -\sum_{\text{Topology}} \alpha_{kj}\left(x_k\left(t-\tau_{kj}\right) - x_j\left(t-\tau_{kj}\right)\right)$$

Delays

Time varying in general

* Olfati-Saber & Murray (2004), Ren (2005), Schollig et al. (2007)

Some Questions

Is the synchronization dynamics stable in presence of delays ?

Effects of topology switches ?

With which delays is stability preserved ?

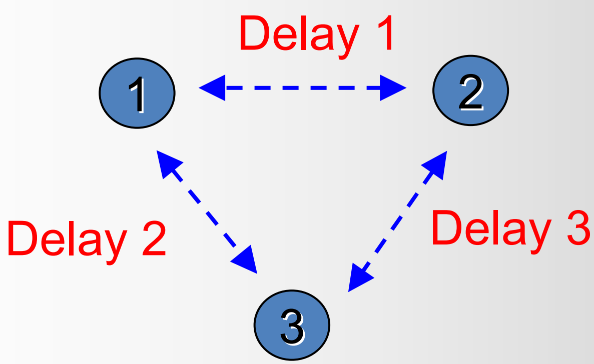
What is the effect of controller gain α_{kj} ?

What is the effect of number of vehicles and the number of delays?

Problem Formulation

Start with a **simpler** problem:

LTI Dynamics with Constant Delays



$$\frac{d\mathbf{x}(t)}{dt} = \sum_{k=1}^{3} \mathbf{A}_k \mathbf{x}(t - \tau_k)$$

Stability Assessment with respect to Three Different Delays ?

Is it **simple**?

Why Delay ? Why Multiple Delays ?

1. Multiple delays represent a more realistic case.
2. Delays may have counter intuitive effects. Different delays may favor/disfavor stability in different ways. One delay may have stabilizing and the other delay may have destabilizing effects in a control system.
3. Ignoring delays or assuming that delays are identical lead to misinterpretation of stability features.

Asymptotic Stability

Start with the Characteristic Equation in Laplace Domain

$$f\left(s,\tau_1,\tau_2,\tau_3\right)=\det\left[s\mathbf{I}-\sum_{k=1}^{3}\mathbf{A}_k\,e^{-\tau_k s}\right]=0$$

For a given set of delays, define the real part of the rightmost roots as

$$\Delta(\overline{\tau}_1,\overline{\tau}_2,\overline{\tau}_3)=\sup\{\Re e(s)\mid f(s,\overline{\tau}_1,\overline{\tau}_2,\overline{\tau}_3)=0,\ s\in\mathbf{C},(\overline{\tau}_1,\overline{\tau}_2,\overline{\tau}_3)\in\Re^{3+}\}$$

It was proven by Richard Datko in 1978 that $\Delta(\tau_1,\tau_2,\tau_3)$ is continuous with respect to delays

Dynamics is stable if and only if $\Delta(\tau_1,\tau_2,\tau_3)<0$

Stability Analysis

Dynamics is stable if and only if $\Delta(\tau_1, \tau_2, \tau_3) < 0$

Stability transitions may occur only when $\Delta(\tau_1, \tau_2, \tau_3) = 0$

Study the characteristic equation for imaginary root solutions

$$f(j\omega, \tau_1, \tau_2, \tau_3) = \det\left[j\omega\mathbf{I} - \sum_{k=1}^{3} \mathbf{A}_k e^{-\tau_k j\omega} \right] = 0$$

An eigenvalue problem with infinitely many solutions

$$j\omega = eig\left(\sum_{k=1}^{3} \mathbf{A}_k e^{-\tau_k j\omega} \right)$$

Methodology for the Stability Analysis

$$\underbrace{\frac{d\mathbf{x}(t)}{dt} = \sum_{k=1}^{3} \mathbf{A}_k \mathbf{x}(t-\tau_k)}$$

To avoid crowding equations, we shall present the method for identical agent dynamics. The method equally works for non-identical agents.

$$\mathbf{A}_1 = \begin{bmatrix} -\alpha & \alpha & 0 \\ \alpha & -\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} -\alpha & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & -\alpha \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & \alpha \\ 0 & \alpha & -\alpha \end{bmatrix}$$

Characteristic Equation for $s = j\,\omega$

$$f(j\omega,\tau_1,\tau_2,\tau_3) = -j\,\omega^3 - 2\,\omega^2\,\alpha\left(e^{-\tau_1\omega j} + e^{-\tau_2\,\omega j} + e^{-\tau_3\omega j}\right) + 3\,j\,\omega\alpha^2\left(e^{-j\omega(\tau_1+\tau_2)} + e^{-j\omega(\tau_1+\tau_3)} + e^{-j\omega(\tau_2+\tau_3)}\right) = 0,$$

Challenges

$$f\left(j\omega,\tau_1,\tau_2,\tau_3\right) = -j\omega^3 - 2\omega^2\alpha\left(e^{-\tau_1\omega j} + e^{-\tau_2\omega j} + e^{-\tau_3\omega j}\right) + 3j\omega\alpha^2\left(e^{-j\omega(\tau_1+\tau_2)} + e^{-j\omega(\tau_1+\tau_3)} + e^{-j\omega(\tau_2+\tau_3)}\right) = 0,$$

1. Visualization of stability in 3D delay space is difficult.
2. Solving delays τ_1 , τ_2 , τ_3 and ω from one equation is difficult.
3. It may be difficult to adapt the existing methods.

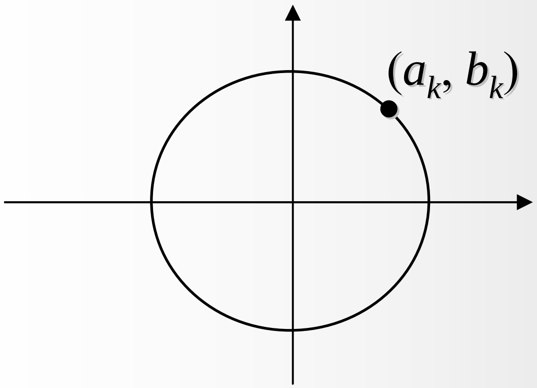
- Obtain 2D cross sectional views of the 3D stability maps.
    - Sweep ω from zero to an upper bound (what is it?).

Stability Analysis

$$f(j\omega,\tau_1,\tau_2,\tau_3)=-j\omega^3-2\omega^2\alpha\left(e^{-\tau_1\omega j}+e^{-\tau_2\omega j}+e^{-\tau_3\omega j}\right)$$
$$+3j\omega\alpha^2\left(e^{-j\omega(\tau_1+\tau_2)}+e^{-j\omega(\tau_1+\tau_3)}+e^{-j\omega(\tau_2+\tau_3)}\right)=0,$$

Given τ_3 and ω solve τ_1, τ_2 from the characteristic equation

$$\left.e^{-\tau_k s}\right|_{s=j\omega}=a_k+jb_k$$



Theorem 1: For a given τ_3 and ω , there exist at most ***eight*** pairs of (τ_1, τ_2) solutions to f(satisfying $0 \le \tau_1\omega < 2\pi$ and $0 \le \tau_2\omega < 2\pi$.

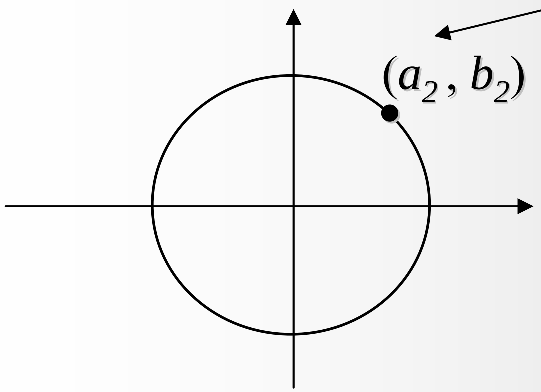
Proof

Theorem 1: For a given τ_3 and ω, there exist at most ***eight*** pairs of (τ_1, τ_2) solutions to f(satisfying $0 \le \tau_1\omega < 2\pi$ and $0 \le \tau_2\omega < 2\pi$.

Proof: Separate the characteristic function into real and imaginary parts:

$$f_R = \mathrm{Re}\left(f\left(i\omega, a_k, b_k\right)\right) \qquad f_I = \mathrm{Im}\left(f\left(i\omega, a_k, b_k\right)\right)$$

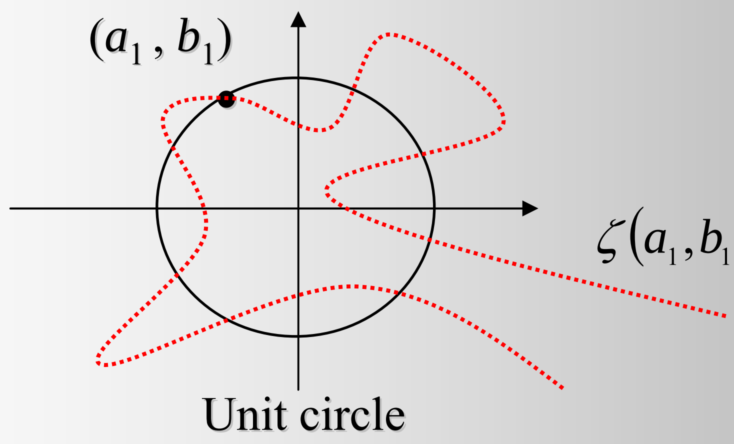
Solve a_2, b_2 pair in terms of a_1 and b_1.



Unit circle

$$\zeta\left(a_1, b_1\right) = \sum_{k=0}^{4} q_k\left(b_1\right) a_1^{\,k} = 0$$

Proof



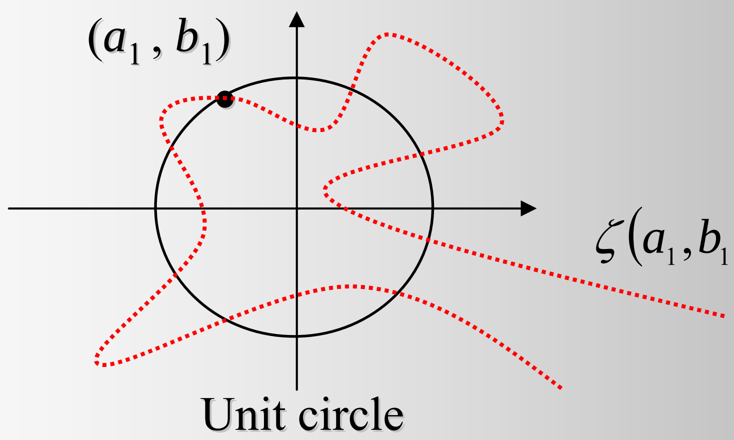
$$\zeta(a_1, b_1) = \sum_{k=0}^{4} q_k(b_1) a_1^{\,k} = 0$$

A 4^{th} power and a 2^{nd} power polynomial have at most $4 \times 2 = 8$ common solutions.

Implementation

Given τ_3 and ω

Find the intersection points using Sylvester's Resultant Matrix



$$\zeta(a_1, b_1) = \sum_{k=0}^{4} q_k(b_1)\, a_1^{\,k} = 0$$

Use those (a_1, b_1) points to back calculate the pairs (a_2, b_2).

Implementation

Use ω and (a_1, b_1) or (a_2, b_2) pairs to compute the delays τ_1 and τ_2

$$\tau_1 = -\frac{1}{\omega}\left[\angle\left(a_1 + j b_1\right)\right]$$

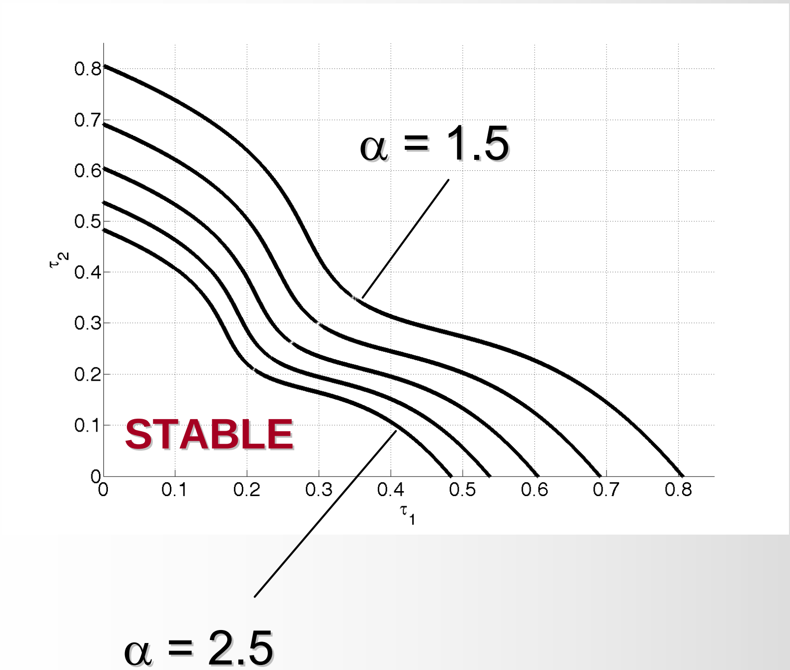
$$\tau_2 = -\frac{1}{\omega}\left[\angle\left(a_2 + j b_2\right)\right]$$

Property: Since the characteristic equation represents a retarded-type dynamics, the upper bound of ω (that can be a solution) is finite.

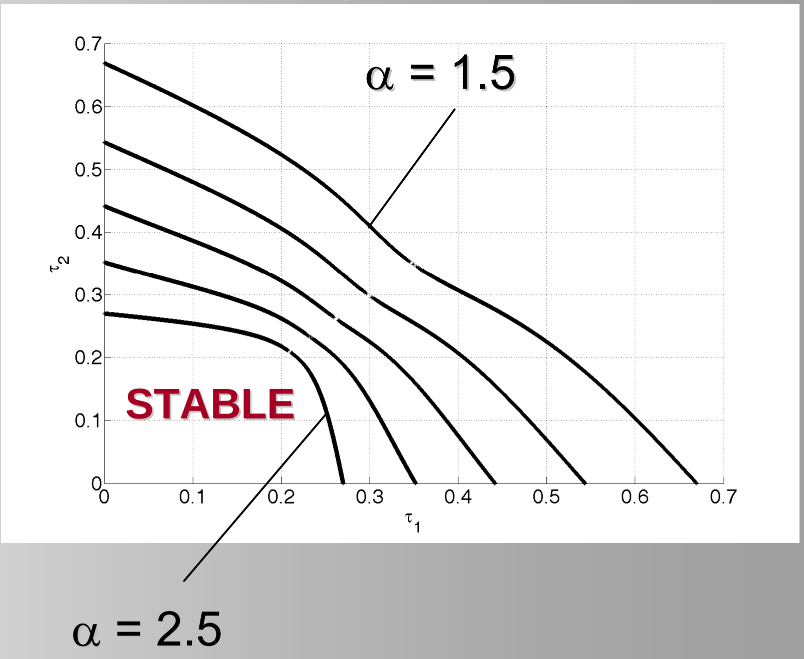
Sweep ω from zero to this upper bound to compute all the delays.

Effects of α

τ_3 = 0

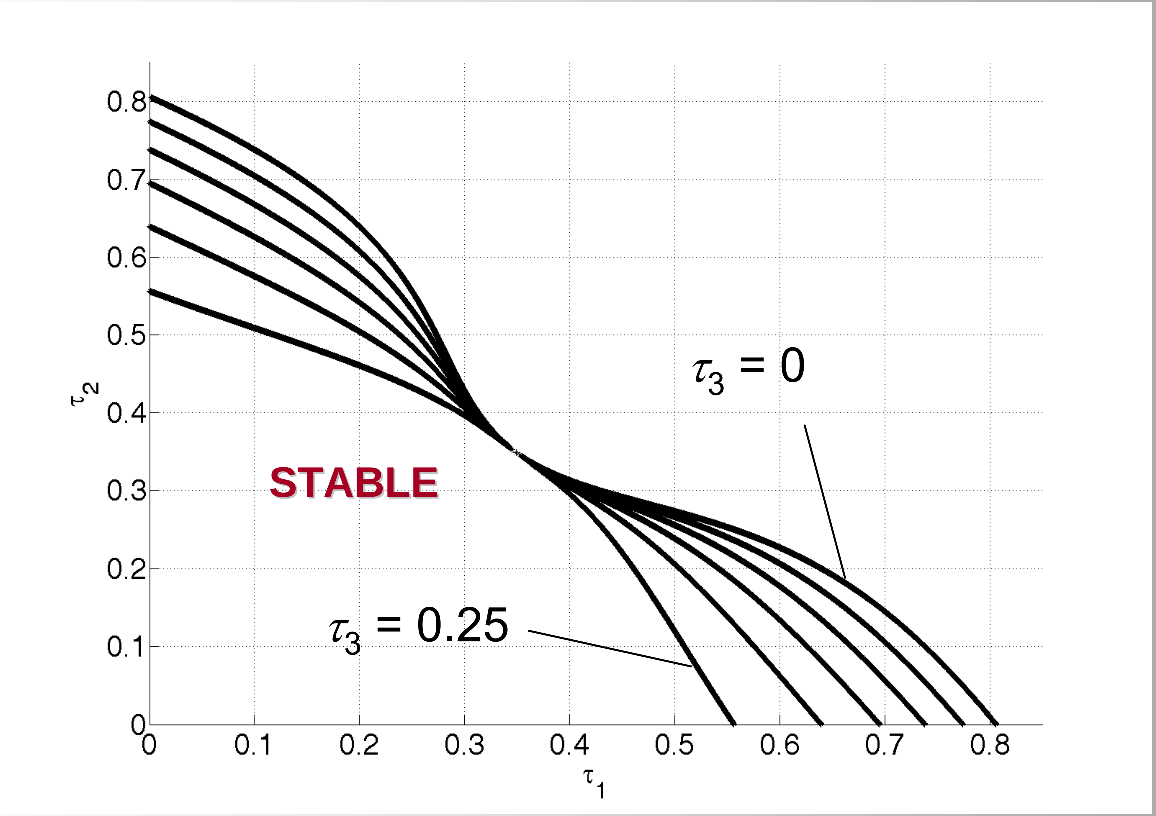


τ_3 = 0.175

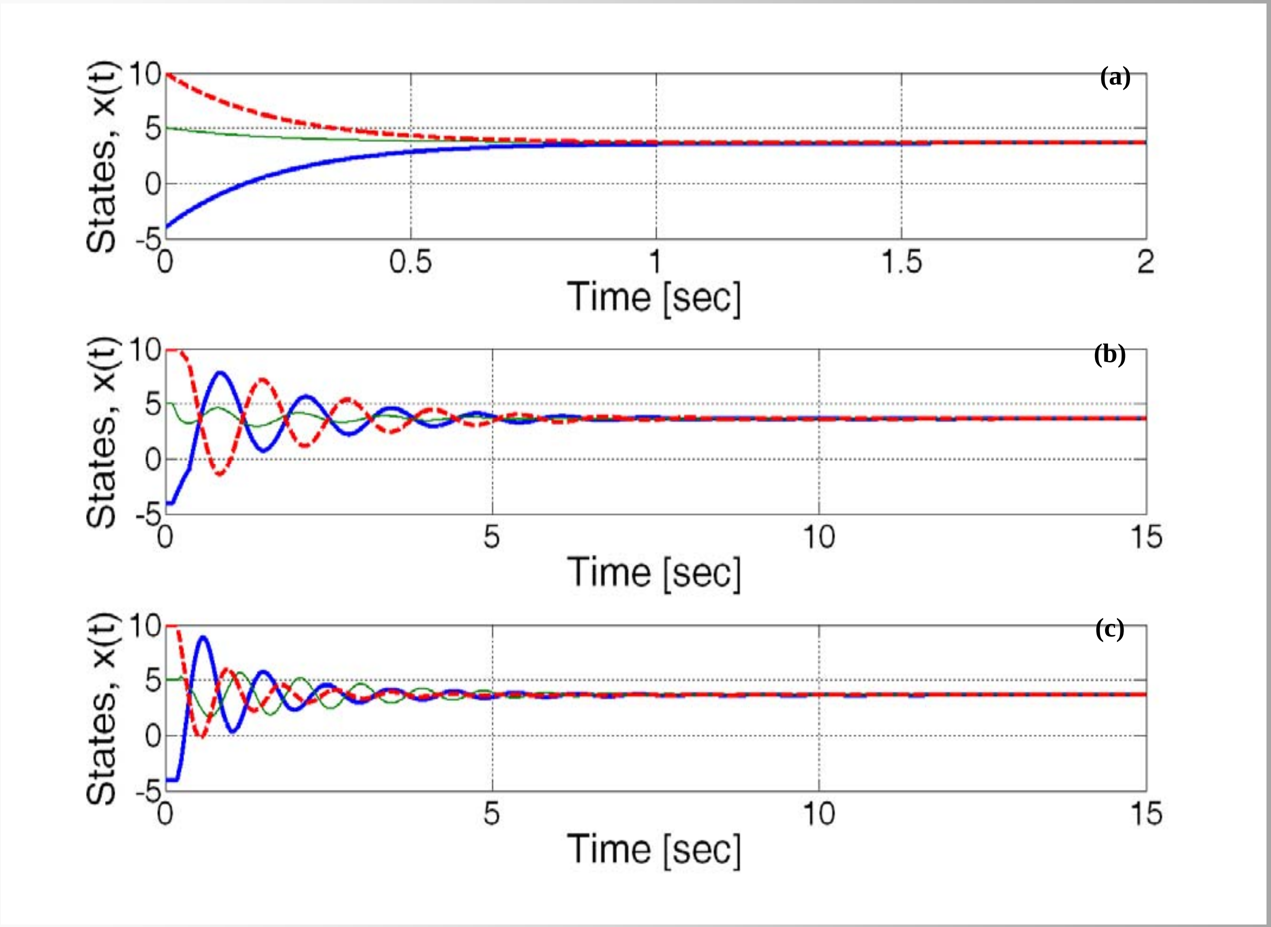


Effects of τ_3

$\alpha = 1.5$



Simulations



SUBPLOT (a): $\alpha = 1.5,\ \tau_1 = \tau_2 = \tau_3 = 0$;
SUBPLOT (b): $\alpha = 1.5,\ \tau_1 = 0.11,\ \tau_2 = 0.36,\ \tau_3 = 0.2$;
SUBPLOT (c): $\alpha = 2.0,\ \tau_1 = 0.24,\ \tau_2 = 0.18,\ \tau_3 = 0.2$.

Conclusions & Future Work

1. Obtaining cross sections of high dimensional stability maps may be a promising venue.
2. Frequency sweeping and geometry arguments can be helpful to assess the stability of systems with large number of delays.
3. Large number of agents with more complicated dynamics are to be considered.
4. Effects of topology are on-going research directions.
5. Effects of topology switches and time-varying nature of the problem are on-going research directions.

THANK YOU !

QUESTIONS ?

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