Stability analysis of delay-parameter space

We consider a stability analysis of a multi-dimensional delay-differential equation

\[ \dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1) + A_2 x(t - \tau_2) + A_3 x(t - \tau_3) \]

where

\[
A_0 = \begin{pmatrix} 3 & 0 \\ -1 & 1.5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & -0.7 \\ 1 & 0.2 \end{pmatrix}, \\
A_2 = \begin{pmatrix} -1.6 & -0.7 \\ 0.8 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -0.7 & 0.2 \\ -0.5 & -2 \end{pmatrix}.
\]

Even though the stability analysis of time-delay systems with multiple delays like this has received a lot of attention in the time-delay community, we believe that it is not widely known that there are methods to plot the region in delay-parameter space in which the system is stable. We list the following publications as examples where a stability analysis is done using these type of plots, sometimes referred to as stability charts, [Beretta, Geometric stability switch criteria in delay differential systems with delay dependent parameters, SIAM J. Math. Anal. 33:1144-1165 2002][Mahaffy et al, A geometric analysis of stability regions for a linear differential equation with two delays, Int. J. Bif. Chaos Appl. Sci. Eng. 5:779-796 1995][Ahlborn et al, Controlling Dynamical Systems using Multiple Feedback Control, Phys. Rev. E 72, 016206 2005][Niculescu et al, On the Stability Crossing Boundaries of Some Delay Systems Modeling Immune Dynamics in Leukemia, Proc 17th Int. Symp. on Math. Theory of Networks and Systems, 2006]. The delay-space stability chart for this equation is visualized in the following figures.
To our knowledge the following types of modern approaches can be used for the computation of delay-space stability charts:

- **Computation of the stability switching curves and surfaces and points where there is an imaginary eigenvalues**
  
  
  
  
  [Jarlebring, Computing the Stability Region in Delay-space of a TDS using Polynomial Eigenproblems, 6th IFAC Workshop on Time-Delay Systems, 2006]
  
  [Ergenc, et al. Kronecker Summation Method and Multiple Delay Systems, 6th IFAC Workshop on Time-Delay Systems 2006],

- **Computation of the rightmost eigenvalues on a grid of points in the stability chart**
  
  
  [Engelborghs, DDE-BIFTOOL A matlab package for bifurcation analysis of delay differential equations, Leuven 2000],

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**Diagram:**

A diagram illustrating the stability region in delay-space is shown. The diagram depicts a graph with axes labeled $\tau_2$ and $\tau_3$, and a boundary between stable and unstable regions. The boundary is marked with a line indicating $\tau_2 = \tau_3 \approx 0.88$.
• Exploitation of the fact that any multiple-delay system can be approximated to arbitrary accuracy with a commensurate system, where exact stability-conditions are known
