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Lateral Vibration of Two Axially Translating Beams Interconnected by a Winkler Foundation

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Transverse vibration of two axially moving beams connected by a Winkler elastic foundation is analyzed analytically. The two beams are tensioned, translating axially with a common constant velocity, simply supported at their ends, and of different materials and geometry. The natural frequencies and associated mode shapes are obtained. The natural frequencies of the system are composed of two infinite sets describing in-phase and out-of-phase vibrations. In case the beams are identical, these modes become synchronous and asynchronous, respectively. Divergence instability occurs at a critical velocity and a critical tension; and, divergence and flutter instabilities coexist at postcritical speeds, and divergence instability takes place precritical tensions. The effects of the mass, flexural rigidity, and axial tension ratios of the two beams are presented. [DOI: 10.1115/1.2732353]

Keywords: double beam system, gyroscopic system, translating beams

1 Background

Web is a generic name for a thin, flexible continuous material found in numerous manufacturing processes. Paper making is one of the oldest of the industries involved with web handling. In the paper-making process, paper fibers are mixed with water and this pulp slurry is sprayed onto a large, flat, fast-moving wire screen, sometimes called the paper cloth. As the wire screen translates along the paper machine, the water drains out, and the fibers bond together. The pulp is pressed between rolls in order to squeeze out more water, and it is further dried by heated rollers. The stiffness of the pulp increases as it is dried along the path of the machine. The paper is eventually wound into a roll and removed from the machine. Vibration problems can arise during transport of the paper-wire system, where excessive vibration could cause the paper to separate from the wire screen prematurely. Dynamic stability of axially moving materials depends on the translation speed. Axially moving materials have been modeled as a string, beam, or plate [1–5]. In this work, the translating wire/paper system is modeled as two translating beams connected by an elastic foundation. The elastic foundation is used to represent the capillary adhesion between the wire and the paper [6].

Vibration of a *translating* web is affected by the convective nature of its transverse acceleration. Dynamics and stability of such gyroscopic systems have been investigated by many investigators. A recent review is given in Ref. [7]. The eigenvalues of general discrete gyroscopic systems are purely imaginary, and the corresponding eigenvectors can be obtained by casting the governing equations in state-space representation, where the orthogonality of the eigenvectors are confirmed, and the solution can be established using the expansion theorem [8]. This also applies to continuous systems [9]. A closed-form solution for axially moving strings and beams, subjected to arbitrary excitations and initial conditions, was given by Wickert and Mote [2]. At supercritical translation speeds, the eigenvalues of the system become complex and divergence and flutter instabilities coexist.

Vibration of a translating string supported by an elastic foundation was studied by Bhat et al. [10], Perkins [11], Wickert [12], and Parker [13]. Presence of the elastic foundation does not change the critical speed predicted by the classical moving threadline theory [11], but the supercritical stability is affected [6]. Vibrations of translating string/beam systems guided by a single spring-loaded guide have been reported in Refs. [14,15], among others.

Use of two (or more) *nontranslating* beams, connected by elastic foundation(s) is common in engineering, and a variety of problems adopt it as a model. The basic model uses a Winkler foundation, in which the beams are connected through closely spaced, but noninterconnected linear springs. The fundamental vibration modes are separated into two groups, where the beams move in phase and out of phase with respect to each other [16–18].

In this paper, the transverse vibrations of two translating, tensioned beams interconnected by an elastic foundation are analyzed. The model represents the coupled behavior of paper translating with the paper cloth during the paper-making process. The effects of damping in the foundation and the viscoelastic nature of paper's elasticity, which has been shown to be important for modeling the dynamics of paper [19,20] are not included in this work but should be the subject of future investigations.

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2 Problem Statement

The system consists of two parallel, slender, prismatic, and homogeneous beams joined by a Winkler foundation of stiffness k. The Winkler foundation is a simplified model for the capillary adhesion forces [6]. Both beams have the same length L between the two supports, simply supported at their ends, axially translating with velocity V, and axially tensioned to p_1 and p_2. The coupled governing equations of the transverse vibrations of the system are derived using Euler-Bernoulli beam theory and can be written as (e.g., [16])

$$E_1I_1\frac{\partial^4 w_1}{\partial x^4}+m_1\left(\frac{\partial^2 w_1}{\partial t^2}+2V\frac{\partial^2 w_1}{\partial x\,\partial t}+V^2\frac{\partial^2 w_1}{\partial x^2}\right)-p_1\frac{\partial^2 w_1}{\partial x^2}+k(w_1-w_2)=f_1 \tag{1}$$

$$E_2I_2\frac{\partial^4 w_2}{\partial x^4}+m_2\left(\frac{\partial^2 w_2}{\partial t^2}+2V\frac{\partial^2 w_2}{\partial x\,\partial t}+V^2\frac{\partial^2 w_2}{\partial x^2}\right)-p_2\frac{\partial^2 w_2}{\partial x^2}+k(w_2-w_1)=f_2 \tag{2}$$

where $w_j=w_j(x,t)$ are the transverse deflections of the two beams $(j=1,2)$, x is the spatial coordinate, t is the time, m_j are the mass per unit length, E_j are the Young's moduli, I_j are the second moment of areas of the beams, k is the stiffness of the Winkler foundation, and f_j are the external forces per unit length. The simple support boundary conditions are

$$w_1(0,t)=\frac{\partial^2 w_1(0,t)}{\partial x^2}=0 \quad \text{and} \quad w_1(L,t)=\frac{\partial^2 w_1(L,t)}{\partial x^2}=0 \tag{3}$$

$$w_2(0,t)=\frac{\partial^2 w_2(0,t)}{\partial x^2}=0 \quad \text{and} \quad w_2(L,t)=\frac{\partial^2 w_2(L,t)}{\partial x^2}=0 \tag{4}$$

The two governing equations can be written in the following nondimensional form:

$$\frac{\partial^4 W_1}{\partial X^4}-(\mu^2-\nu^2)\frac{\partial^2 W_1}{\partial X^2}+2\nu\frac{\partial^2 W_1}{\partial X\,\partial T}+\frac{\partial^2 W_1}{\partial T^2}+K(W_1-W_2)=F_1 \tag{5}$$

$$\frac{\partial^4 W_2}{\partial X^4}-\left[\left(\frac{R_s}{R_p}\right)\mu^2-\left(\frac{R_s}{R_m}\right)\nu^2\right]\frac{\partial^2 W_2}{\partial X^2}+2\nu\left(\frac{R_s}{R_m}\right)\frac{\partial^2 W_2}{\partial T\,\partial X}+\left(\frac{R_s}{R_m}\right)\frac{\partial^2 W_2}{\partial T^2}+R_sK(W_2-W_1)=F_2 \tag{6}$$

with the nondimensional variables defined as

$$X=\frac{x}{L};\quad W_j=\frac{w_j}{L};\quad T=t\left(\frac{E_1I_1}{m_1L^4}\right)^{1/2};\quad \mu^2=\frac{p_1L^2}{E_1I_1};$$
$$K=\frac{kL^4}{E_1I_1};\quad \nu=V\left(\frac{m_1L^2}{E_1I_1}\right)^{1/2};\quad R_m=\frac{m_1}{m_2};\quad R_s=\frac{E_1I_1}{E_2I_2};\quad R_p=\frac{p_1}{p_2};$$
$$R_f=\frac{f_1}{f_2};\quad F_1=\frac{f_1L^3}{E_1I_1};\quad F_2=\frac{R_s}{R_f}F_1;\quad \bar{\lambda}=\frac{\lambda}{\left(\frac{E_1I_1}{m_1L^4}\right)^{1/2}} \tag{7}$$

where T is the nondimensional time, μ^2 is the nondimensional tension parameter, K is the nondimensional elastic foundation stiffness, R_m is the mass ratio, R_s is the flexural stiffness ratio, R_p is the axial load ratio of the beams, $\bar{\lambda}$ is the nondimensional eigenvalue, ν is the nondimensional axial translation speed, R_f is the external force ratio, and F_i are the nondimensional external forces. The nondimensional forms of the boundary conditions become

$$W_1(0,T)=\frac{\partial^2 W_1(0,T)}{\partial X^2}=0 \quad \text{and} \quad W_1(1,T)=\frac{\partial^2 W_1(1,T)}{\partial X^2}=0 \tag{8}$$

$$W_2(0,T)=\frac{\partial^2 W_2(0,T)}{\partial X^2}=0 \quad \text{and} \quad W_2(1,T)=\frac{\partial^2 W_2(1,T)}{\partial X^2}=0 \tag{9}$$

3 Natural Frequency Analysis

In order to obtain the natural frequencies and the mode shapes of the system, the homogenous form of Eqs. (5) and (6) are considered. The response of beam 2 is expressed in terms of the response of the beam 1, from Eq. (5) as follows:

$$W_2=\frac{1}{K}\left[\frac{\partial^4 W_1}{\partial X^4}-(\mu^2-\nu^2)\frac{\partial^2 W_1}{\partial X^2}+2\nu\frac{\partial^2 W_1}{\partial X\,\partial T}+\frac{\partial^2 W_1}{\partial T^2}+KW_1\right] \tag{10}$$

Equations (5) and (6) are then combined into a single eighth-order partial differential equation

$$\frac{\partial^8 W1}{\partial X^8}+A_1\frac{\partial^6 W_1}{\partial X^6}+A_2\frac{\partial^6 W_1}{\partial X^5\,\partial T}+A_3\frac{\partial^6 W_1}{\partial X^4\,\partial T^2}+A_4\frac{\partial^4 W_1}{\partial X^4}+A_5\frac{\partial^4 W_1}{\partial X^3\,\partial T}+A_6\frac{\partial^4 W_1}{\partial X^2\,\partial T^2}+A_7\frac{\partial^4 W_1}{\partial X\,\partial T^3}+A_8\frac{\partial^4 W_1}{\partial T^4}+A_9\frac{\partial^2 W_1}{\partial X^2}+A_{10}\frac{\partial^2 W_1}{\partial T^2}+A_{11}\frac{\partial W_1}{\partial T}=0 \tag{11}$$

The constant coefficients A_n $(n=1,\ldots,11)$, given in the Appendix, depend on the system parameters. The eigenfunction for beam 1 is shown to be [1]

$$\hat{\phi}_1(X)=\sum_{k=1}^{8}c_k e^{i\gamma_k X} \tag{12}$$

where c_k are constant coefficients and γ_k are the roots of the characteristic equation of Eq. (11). This characteristic equation is obtained by substituting $e^{i\gamma_k X}e^{\bar{\lambda}T}$ into Eq. (11). The eigenfunction for beam 2 is found by substituting Eq. (12) into Eq. (10), and becomes

$$\hat{\phi}_2(X)=\sum_{k=1}^{8}B_k c_k e^{i\gamma_k X} \tag{13}$$

with

$$B_k=\frac{1}{K}\left[\gamma_k^4+(\nu^2-\mu^2)\gamma_k^2+2\nu i\gamma_k\bar{\lambda}+\bar{\lambda}^2+K\right] \tag{14}$$

In order to obtain the eigenvalues for the double-beam system, boundary conditions, in Eqs. (8) and (9) are evaluated using Eqs. (12) and (13). This results in eight homogeneous algebraic equations, which are represented in matrix form as

$$\boldsymbol{D}(\bar{\lambda})\cdot\boldsymbol{c}=0 \tag{15}$$

where $\boldsymbol{c}=\{c_1,c_2,c_3,\ldots,c_8\}^T$ is the coefficient vector and $\boldsymbol{D}(\bar{\lambda})$ is the matrix of coefficients. In order to have a nontrivial solution, the determinant of matrix $|\boldsymbol{D}(\bar{\lambda})|$ must be zero. This gives the characteristic equation of the system. The natural frequencies are determined from the solution of the characteristic equation. The mode shapes are then calculated from Eqs. (12) and (13) and normalized using the real parts of the complex mode shapes with respect to the symmetric matrix operator $\boldsymbol{A}$ as $\langle A\phi_m^R,\phi_n^R\rangle=\delta_{mn}$ [1].

4 Results and Discussion

4.1 Mode Shapes and Natural Frequencies. The natural frequencies of the translating double-beam system are divided into two fundamental odd and even sets $\omega_{(2n-1)}$ and $\omega_{(2n)}$, for $n=1,2,\ldots$ The real and the imaginary parts of the odd-numbered mode shapes display in-phase deflections, and those of the even-

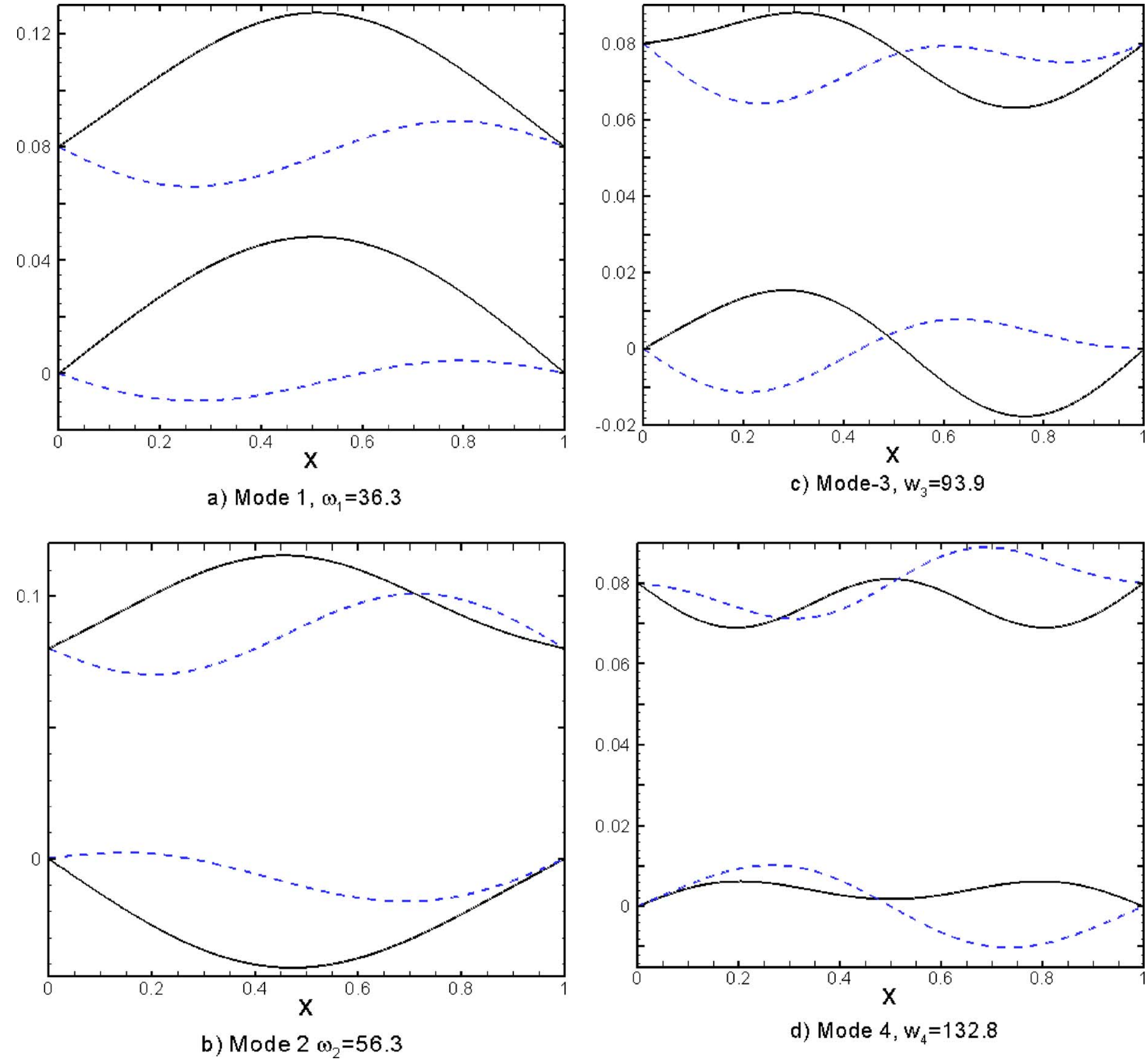

Fig. 1 First four complex mode shapes for a traveling system with nonequal mass densities, R_m=0.6; the real (solid line) and imaginary parts (dashed line) for K=100, ν=5, μ=10, and R_p=R_s=1

numbered mode shapes display out-of-phase deflections. When the two beams are identical in all respects, the in-phase mode shapes are anti-symmetric with respect to the midheight, in the thickness direction, while the out-of-phase mode shapes are symmetric. Thus, the in-phase modes of identical beams do not deflect the elastic foundation, whereas the opposite is true for the out-of-phase modes. The presence of in-phase and out-of-phase modes is the result of the coupling of the two beams by the Winkler foundation, and it is observed for nontranslating beam systems (e.g., [16,18]).

Figure 1 shows the first four modes for the mass ratio $R_m = 0.6$. The symmetry and anti-symmetry of the in-phase and out-of-phase mode shapes, respectively, are distorted as the material properties are not identical. This effect is observed for other R_m values, as well as R_p and R_s values, and parallelism of the modes deteriorate further with decreasing values of R_m, R_p, and R_s. For nonidentical traveling beams, both groups of modes shapes are always affected by the coupling.

4.2 Effect of Translation Speed on Natural Frequencies. The effect of the nondimensional translation speed ν on the eigenvalues $\bar{\lambda}$ of the two identical beams described above are shown in Fig. 2. Similar to a single, tensioned, simply supported, axially moving beam analyzed in Refs. [2,21],[2] the first natural frequency vanishes at the critical speed $\nu_c=(\mu^2+\pi^2)^{1/2}$. The eigenvalues for the double-beam system are purely imaginary for $\nu<\nu_c$ but become complex for larger values of ν. The first natural frequency is not affected by the presence of the second beam. This result is expected because the odd-numbered natural frequencies (in phase) are not affected by the presence of the Winkler foundation for identical beams. Hence, the onset of divergence instability for the double-beam system analyzed here is identical to the case of the single beam, and the elastic stiffness does not alter the divergence instability.

The first type of instability to occur for both in-phase and out-of-phase modes is the divergence instability with $\mathrm{Im}(\bar{\lambda})=0$ and $\mathrm{Re}(\bar{\lambda})>0$. For faster translation speeds, flutter occurs for both beams. This behavior is not altered as compared to the single-beam case. Moreover, the out-of-phase frequency spectrum behaves qualitatively, the same way as the in-phase part. The in-

[2]Note that the critical speed expression given in Ref. 21 for a traveling Timoshenko beam can be shown to be identical to ν_c given by [2] after letting rotary inertia, $\rho I=0$, and shear stiffness $G\rightarrow\infty$.

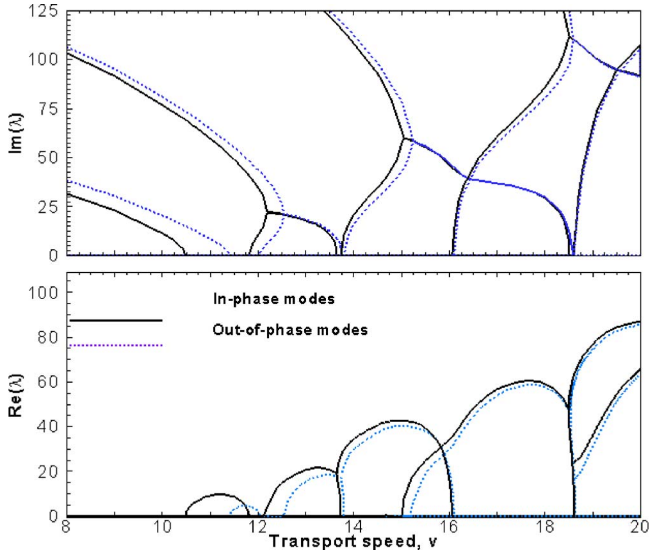

Fig. 2 Imaginary and real parts of the nondimensional natural frequencies of a system with identical properties, as a function of translation speed v, K=100, μ=10, and $R_m=R_s=R_p=1$

phase and out-of-phase frequency curves cluster near the regions of flutter instability. Therefore, both in-phase and out-of-phase modes would participate in the system's response if it is excited near these frequencies.

4.3 Effect of Foundation Stiffness on Natural Frequencies. For coupled beam systems with identical properties, the foundation stiffness K has no effect on the in-phase modes but does affect the out-of-phase modes [1,6]. On the other hand, rendering R_m, R_s, and R_p values <1 makes the in-phase frequencies become slightly dependent on K; and increases the out-of-phase frequencies [1,6]. The increase of the in-phase frequencies, in this case, is due to the coupling between the beams, which is felt by both types of modes when beams are not identical.

4.4 Effect of Axial Tension on Natural Frequencies. Next, the effects of the non dimensional tension parameter μ on the natural frequencies of the system are investigated in the range $0 \leq \mu \leq 10$. The results are shown in Fig. 3, where the natural frequencies are presented for different values of R_m, R_s, and R_p for the translation velocity of v=5. As before, the base values are chosen for two identical beams, as $R_m=R_s=R_p=1$, K=100, unless otherwise noted.

The effect of applying different nondimensional tension (μ) values on the natural frequencies is shown in Fig. 3(*a*), for the case of identical beams. Figure 3(*a*) shows that frequencies increase with increasing μ, while the values of the natural frequencies for the in-phase and out-of-phase modes remain relatively close to each other. An exception occurs near the *minimum stabilizing tension* μ_{cr}=3.9, where $\mathrm{Im}(\bar{\lambda})$ vanishes and the $\mathrm{Re}(\bar{\lambda})$ becomes positive. The system displays divergence instability when $\mu<\mu_{cr}$. Note that this μ_{cr} value is valid for the nondimensional speed value of v=5.

The effect of nonidentical beam mass is investigated by letting the mass ratio R_m=0.9. Results are presented in Fig. 3(*b*), which shows that the frequencies of the in-phase modes (ω_1, ω_3, ω_5) decrease only slightly, whereas frequencies of the out-of-phase modes (ω_2, ω_4, ω_6) remain close to the identical beam case given in Fig. 3(*a*).

The effect of bending rigidity ratio, R_s=0.1, is shown in Fig. 3(*c*). Figure 3(*c*) shows that the first two in-phase-mode frequencies (ω_1, ω_3) are not affected significantly by the reduction in the nondimensional bending rigidity R_s; however, the third in-phase-

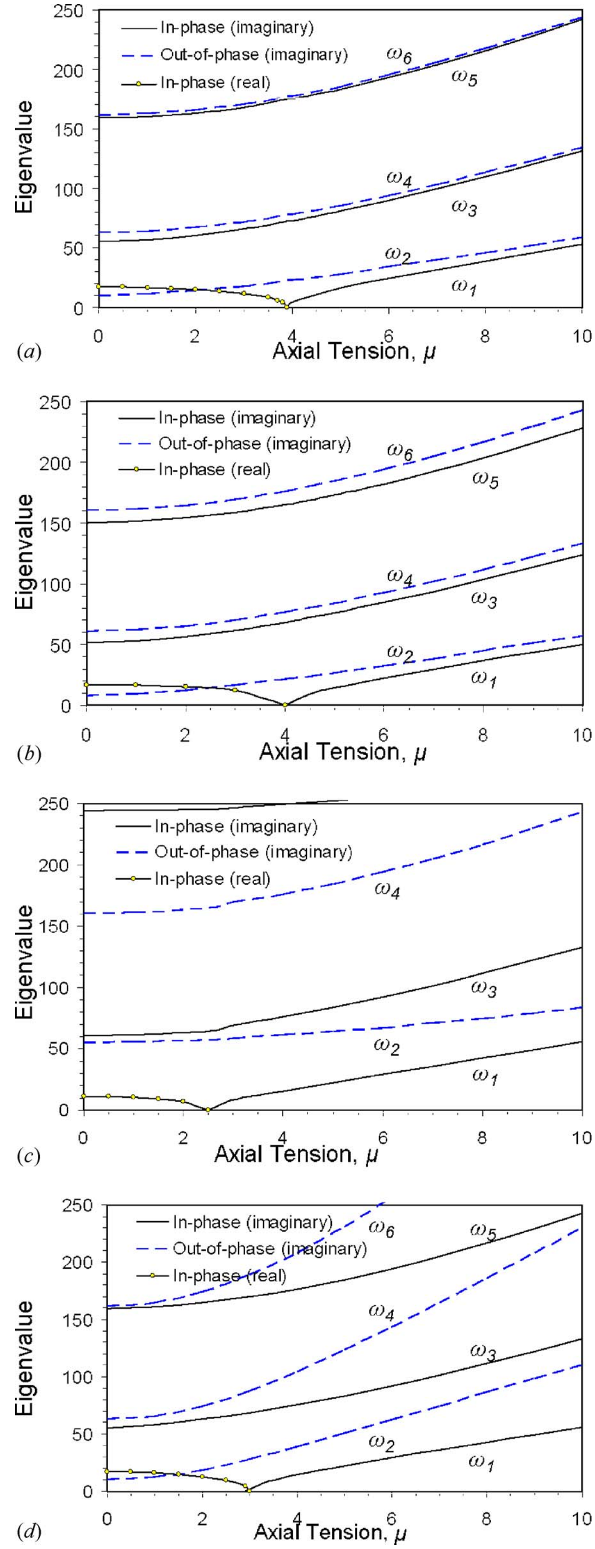

Fig. 3 Nondimensional natural frequencies as a function of the nondimensional axial tension parameter μ for K=100 and v=5

mode frequency (ω_5) and the out-of-phase-mode frequencies increase, with respect to the case of identical beams. The minimum stabilizing tension is approximately $\mu_{cr}\cong 2.5$.

The effect of nonequal tension distribution between the two beams is shown in Fig. 3(*d*), where the axial load ratio is R_p

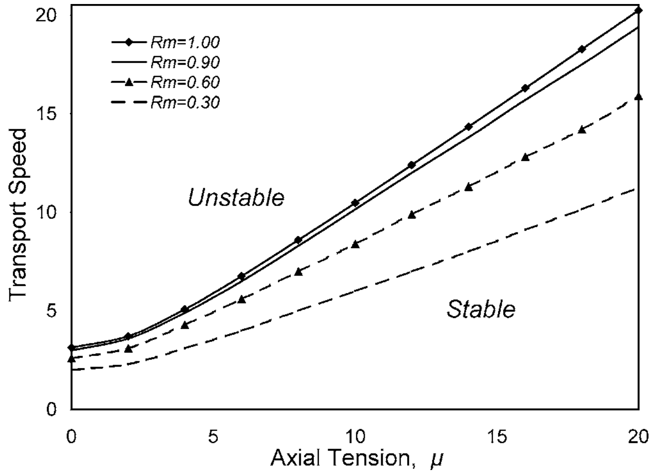

Fig. 4 Nondimensional transport speed ν as a function of nondimensional axial tension parameter μ for $K=100$ and $R_s=R_p=1$ and different mass ratios R_m

$=0.3$. The in-phase-mode frequencies are not affected significantly, whereas the out-of-phase mode frequencies show a large increase for the higher μ values with respect to the identical beam case (Fig. 3(*a*)). The minimum stabilizing tension is approximately $\mu_{cr} \cong 3$.

In summary, by increasing the nondimensional tension μ, the overall stiffness of the system increases, causing the natural frequencies of the in-phase and out-of-phase modes to increase. Increasing the mass, bending rigidity, and tension of one of the beams affects the out-of-phase modes and causes their natural frequencies to increase. Out-of-phase modes are affected more strongly by increasing tension because the system represented by these modes is more strongly subjected to the effect of the foundation stiffness.

4.5 Effect of Axial Tension on Critical Speed and Stability. As discussed in Sec. 4.2, the critical speed ν_c depends on the axial tension. Stability regions as a function of nondimensional translation speed ν and nondimensional tension μ are presented in Fig. 4, with different mass ratios ($R_m=1$, 0.9, 0.6, 0.3) and $K=100$, $R_s=R_p=1$. It is seen that lowering the mass ratio R_m reduces the stable domain. This is expected because lowering the mass ratio causes the mass of one of the beams to increase. On the other hand, the bending rigidity R_s and axial tension R_p ratios have a small effect on the stability regions [6].

5 Conclusions

The natural frequencies of an elastically connected, axially loaded, simply supported, axially translating double-beam system are composed of two infinite sets, where the odd-numbered frequencies show in-phase and even-numbered frequencies show out-of-phase vibrations. Divergence instability occurs at the critical speed ν_{cr}, and flutter and divergence instabilities coexist in postcritical speeds, with frequencies of the both modes clustering near flutter regions. In the case of identical beams, the presence of the elastic foundation does not affect the critical speed. Lowering the tension of one of the beams causes divergence instability at the minimum stabilizing tension μ_{cr}. In cases where the stiffness or axial loading ratios between the beams are nonidentical, the out-of-phase frequencies are affected more significantly. The stability regions are obtained as a function of axial tension and critical speed for different mass ratios. It is found that the critical speeds could become significantly lower for low mass ratios.

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Appendix

The coefficients of Eq. (11) are

$$A_1 = \left(1 + \frac{R_s}{R_m}\right)\nu^2 - \left(1 + \frac{R_s}{R_p}\right)\mu^2 \qquad (\text{A}1a)$$

$$A_2 = 2\nu\left(1 + \frac{R_s}{R_m}\right) \qquad (\text{A}1b)$$

$$A_3 = \left(1 + \frac{R_s}{R_m}\right) \qquad (\text{A}1c)$$

$$A_4 = K(1 + R_s) + \left(\frac{R_s}{R_m}\nu^2 - \frac{R_s}{R_p}\mu^2\right)(\nu^2 - \mu^2) \qquad (\text{A}1d)$$

$$A_5 = 2\nu\left(\frac{R_s}{R_m}\nu^2 - \frac{R_s}{R_p}\mu^2\right) + 2\nu\left(\frac{R_s}{R_m}\right)(\nu^2 - \mu^2) \qquad (\text{A}1e)$$

$$A_6 = \left(\frac{R_s}{R_m}\nu^2 - \frac{R_s}{R_p}\mu^2\right) + \left(\frac{R_s}{R_m}\right)(\nu^2 - \mu^2) + (2\nu)^2\left(\frac{R_s}{R_m}\right) \qquad (\text{A}1f)$$

$$A_7 = 4\nu\left(\frac{R_s}{R_m}\right) \qquad (\text{A}1g)$$

$$A_8 = \left(\frac{R_s}{R_m}\right) \qquad (\text{A}1h)$$

$$A_9 = K\left(\frac{R_s}{R_m}\nu^2 - \frac{R_s}{R_p}\mu^2\right) + R_sK(\nu^2 - \mu^2) \qquad (\text{A}1i)$$

$$A_{10} = R_sK\left(1 + \frac{1}{R_m}\right) \qquad (\text{A}1j)$$

$$A_{11} = 2\nu R_sK\left(1 + \frac{1}{R_m}\right) \qquad (\text{A}1k)$$

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