Transverse vibration of two axially moving beams connected by a Winkler elastic foundation is analyzed analytically. The system is a model of paper and paper-cloth (wire-screen) used in paper making. The two beams are tensioned, translating axially with a common constant velocity, simply supported at their ends, and of different materials and geometry. Due to the effect of translation, the dynamics of the system displays gyroscopic motion. The Euler-Bernoulli beam theory is used to model the deflections, and the governing equations are expressed in the canonical state form. The natural frequencies and associated mode shapes are obtained. It is found that the natural frequencies of the system are composed of two infinite sets describing in-phase and out-of-phase vibrations. In case the beams are identical, these modes become synchronous and asynchronous, respectively. Divergence instability occurs at the critical velocity; and, the frequency-velocity relationship is similar to that of a single traveling beam. The effects of the mass, flexural rigidity, and axial tension ratios of the two beams, as well as the effects of the elastic foundation stiffness are investigated.

Keywords: Double beam system, Gyroscopic system, Axially tensioned beams, Translating beams.

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1. INTRODUCTION

Axially moving materials are found in many engineering applications such as magnetic tape systems, fiber winders, power transmission belts, textile manufacturing and paper-web handling machinery. Axially moving materials typically are modeled as a string or as an Euler-Bernoulli beam [2][3]. *Web* is a generic name used for thin, flexible continuous materials such as magnetic tapes and papers. Paper making is one of the oldest of the industries involved with web handling, with more than a century of history. In the papermaking process, paper fibers are mixed with water, and this pulp slurry is sprayed onto a large, flat, fast-moving wire-screen, sometimes called the paper-cloth. As the wire-screen translates along the paper machine, the water drains out, and the fibers bond together. The paper web is pressed between rolls in order to squeeze out more water and it is further dried by heated rollers. The stiffness of paper increases as it is dried along the path of the machine. The paper is eventually rolled and removed from the machine. Vibration problems can arise during transport of the paper-wire system, where excessive vibration could cause the paper to separate from the wire-screen prematurely. In this work the translating wire/paper system is modeled as two translating beams, connected by an elastic foundation. The elastic foundation is used to represent the capillary-adhesion between the wire and the paper.

Vibration of systems that can be modeled as a *translating* single beam or string have been studied extensively. Transverse vibrations of translating continua are affected not only by the *local* acceleration, but also by the *Coriolis* and the *centripetal* accelerations, which arise due to convection of the material. Dynamics and stability of such gyroscopic systems have been investigated by many investigators; in depth reviews of the literature are given in references [4], [5], and [6]. The eigenvalues of general discrete gyroscopic systems are purely imaginary, and
the corresponding eigenvectors can be obtained by casting the governing equations in state space representation, where the orthogonality of the eigenvectors are confirmed, and the solution can be established using the expansion theorem [7]. This also applies to continuous systems [8]. A closed form solution for the general axially moving continua problems, subjected to arbitrary excitations and initial conditions was given by Wickert and Mote [9]. It was shown that at supercritical critical translation speeds the eigenvalues of the system become imaginary and divergence and flutter instabilities co-exist.

Vibration of a translating string supported by an elastic foundation was studied by Bhat et al. [10], Perkins [11], Wickert [12] and Parker [13]. Bhat et al. formulated the problem to include the non-linear deformation of the string arising from large amplitude oscillations, as well as the gyroscopic terms. They obtained the transient solution of the system by using numerical techniques. They showed that the response of the translating system is not periodic and the system becomes unstable when the translation velocity approaches the wave speed in the belt [10]. Perkins analyzed the dynamics of a traveling string on an elastic foundation using analytical methods, where he showed that the natural frequencies and mode shapes of this system depend on the tension, mass, translation velocity and the foundation stiffness; however, the presence of the elastic foundation does not change the critical speed predicted by the classical moving threadline theory [11]. Wickert [12] showed that a closed form solution of this system exists with standard solution form of a continuous gyroscopic system [8][9]. Parker pointed out that the elastic foundation does not alter the lowest critical speed, but the supercritical stability is changed by the presence of an elastic foundation [6]. Vibrations of translating string/beam systems guided by a single spring loaded guide have been reported in references [14] - [18] among others.
Use of two (or more) *non-translating* beams, connected by elastic foundation(s) is common in engineering, and a variety of problems adopt it as a model. The basic model uses a Winkler foundation, in which the beams are connected through closely spaced, but non-interconnected linear springs, which is defined by the foundation stiffness $k$. A considerable number of theoretical and experimental works on the transverse vibrations of such systems has been performed. The work of Seelig and Hoppman appears to be the first study where the coupled vibration of two beams connected by an elastic foundation was investigated [19]. They obtained the natural frequencies and associated mode shapes for various supports conditions; and, showed that fundamental vibration modes are separated into two groups where the beams move in-phase and out-of-phase with respect to each other. Kessl investigated the effect of a cyclic load on a elastically connected, simply supported double-beam system [20]. He and Rao used the energy method and Hamilton’s principal to derive the governing equations, and the corresponding essential and natural boundary conditions, for multi-span beams connected by an elastic (adhesive) layer. They presented an analytical solution for the coupled transverse and longitudinal vibration of the system with arbitrary boundary conditions [21]. Kukla obtained analytical solution of two axially loaded beams interconnected by multiple discrete springs, by developing a Green’s function [22]. Vu *et al.* developed a closed form solution for the vibration of a double beam system connected by distributed springs and dashpots representing a viscoelastic material [23]. Analytical solutions were obtained by decoupling the governing equations through a change of the variables, which is applicable only when the bending rigidity and the boundary conditions of the beams are identical. They showed that presence of damping suppresses the out-of-phase modes, as energy dissipation depends on relative motion of the two beams. Oniszczuk presented the free and forced vibration analysis of two identical strings,
interconnected by distributed springs and dampers by decoupling the governing equations [24]. Oniszczuk also analyzed the undamped free vibrations of an elastically connected, simply-supported double beam system by using the classical modal expansion method [25]. He obtained an analytical solution, by using a mode shape function of a single simply-supported beam, and determined the natural frequencies and the complete dynamic response.

In this paper the transverse vibrations of two translating, tensioned beams interconnected by an elastic foundation are analyzed. The model represents the coupled behavior of paper translating with the paper cloth during paper making process.

2. PROBLEM STATEMENT

The system shown in Fig. 1 consists of two parallel, slender, prismatic and homogeneous beams, joined by a Winkler foundation of stiffness $k$. The Winkler foundation is a simplified model for the capillary adhesion forces as discussed in Appendix A. Both beams have the same length $L$ between the two supports, simply supported at their ends, axially translating with velocity $V$, and axially tensioned to $p_1$ and $p_2$ as shown. The coupled governing equations of the transverse vibrations of the system are derived using Bernoulli-Euler beam theory and can be written as (e.g. [19]):

$$E_1 I_1 \frac{\partial^4 w_1}{\partial x^4} + m_1 \left( \frac{\partial^2 w_1}{\partial t^2} + 2V \frac{\partial^2 w_1}{\partial x \partial t} + V^2 \frac{\partial^2 w_1}{\partial x^2} \right) - p_1 \frac{\partial^2 w_1}{\partial x^2} + k (w_1 - w_2) = f_1$$  (1)

$$E_2 I_2 \frac{\partial^4 w_2}{\partial x^4} + m_2 \left( \frac{\partial^2 w_2}{\partial t^2} + 2V \frac{\partial^2 w_2}{\partial x \partial t} + V^2 \frac{\partial^2 w_2}{\partial x^2} \right) - p_2 \frac{\partial^2 w_2}{\partial x^2} + k (w_2 - w_1) = f_2$$  (2)

where $w_j = w_j(x,t)$ are the transverse deflections of the two beams ($j = 1, 2$), $x$ is the spatial coordinate, $t$ is the time, $m_j$ are the mass per unit length, $E_j$ are the Young’s moduli, $I_j$ are the
second moment of areas, $k$ is the stiffness of the Winkler foundation, $V$ is the axial translation speed of the beams and $f_j$ are the external forces per unit length. The local acceleration of the beams is represented by the $\frac{\partial^2 w_j}{\partial t^2}$ term; the Coriolis acceleration is represented by the $2V \frac{\partial^2 w_j}{\partial x \partial t}$ term; and, the centrifugal acceleration is represented by the $V^2 \frac{\partial^2 w_j}{\partial x^2}$ term.

Both beams are assumed to be simply supported at their ends $x = 0$ and $x = L$. The simple support boundary conditions are:

$$w_1(0,t) = \frac{\partial^2 w_1(0,t)}{\partial x^2} = 0 \quad \text{and} \quad w_1(L,t) = \frac{\partial^2 w_1(L,t)}{\partial x^2} = 0$$

(3)

$$w_2(0,t) = \frac{\partial^2 w_2(0,t)}{\partial x^2} = 0 \quad \text{and} \quad w_2(L,t) = \frac{\partial^2 w_2(L,t)}{\partial x^2} = 0$$

(4)

The two governing equations can be written in the following non-dimensional form:

$$\frac{\partial^4 W_1}{\partial X^4} - (\mu^2 - \nu^2) \frac{\partial^2 W_1}{\partial X^2} + 2\nu \frac{\partial^2 W_1}{\partial X \partial T} + \frac{\partial^2 W_1}{\partial T^2} + K(W_j - W_s) = F_1$$

(5)

$$\frac{\partial^4 W_2}{\partial X^4} - \left(\frac{R_p}{R_m}\right) \mu^2 - \left(\frac{R_m}{R_p}\right) \nu^2 \right) \frac{\partial^2 W_2}{\partial X^2} + 2\nu \left(\frac{R_p}{R_m}\right) \frac{\partial^2 W_2}{\partial T \partial X} \left(\frac{R_m}{R_p}\right) \frac{\partial^2 W_2}{\partial T^2} + R_s K(W_s - W_j) = F_2$$

(6)

with the non-dimensional variables defined as:

$$X = \frac{x}{L}, \quad W_j = \frac{W_j}{L}, \quad T = t \left(\frac{E_1 I_1}{m_1 L^4}\right)^{1/2}, \quad \mu^2 = \frac{p_1 L^2}{E_1 I_1}, \quad K = \frac{k L^4}{E_1 I_1}, \quad \nu = V \left(\frac{m_1 L^2}{E_1 I_1}\right)^{1/2},$$

$$R_m = \frac{m_1}{m_2}, \quad R_s = \frac{E_1 I_1}{E_2 I_2}, \quad R_p = \frac{p_1}{p_2}, \quad R_f = \frac{f_1}{f_2}, \quad F_1 = \frac{f_1 L^3}{E_1 I_1}, \quad F_2 = \frac{R_s}{R_f} F_1, \quad \lambda = \frac{\lambda}{\left(\frac{E_1 I_1}{m_1 L^4}\right)^{1/2}}$$

(7)

where $T$ is the non-dimensional time, $\mu^2$ is the non-dimensional tension parameter, $K$ is the non-dimensional elastic foundation stiffness, $R_m$ is the mass ratio, $R_s$ is the flexural stiffness ratio, $R_p$ is the axial load ratio of the beams, $\lambda$ is the non-dimensional complex natural frequency, $\nu$ is the
non-dimensional axial translation speed, \( R_f \) is the external force ratio, and \( F_i \) are the non-dimensional external forces. The non-dimensional forms of the boundary conditions become:

\[
W_1(0, T) = \frac{\partial^2 W_1(0, T)}{\partial X^2} = 0 \quad \text{and} \quad W_1(1, T) = \frac{\partial^2 W_1(1, T)}{\partial X^2} = 0
\]  

(8)

\[
W_2(0, T) = \frac{\partial^2 W_2(0, T)}{\partial X^2} = 0 \quad \text{and} \quad W_2(1, T) = \frac{\partial^2 W_2(1, T)}{\partial X^2} = 0
\]  

(9)

3. SOLUTION METHOD

The system of equations given by (5) and (6) can be written in the form of a system of second order differential equations as:

\[
MW_{,,TT} + GW_{,,T} + K^*W = f
\]  

(10)

where a subscripted comma ,,T indicates partial differentiation, and

\[
W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \quad f = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ 0 & \left( \frac{R_s}{R_m} \right)^2 \end{bmatrix}
\]  

(11)

\[
K^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial^4}{\partial X^4} + \begin{bmatrix} (v^2 - \mu^2) & 0 \\ 0 & \left( \frac{R_s}{R_m} \right) v^2 - \left( \frac{R_s}{R_p} \right) \mu^2 \end{bmatrix} \frac{\partial^2}{\partial X^2} + \begin{bmatrix} 1 & -I \\ -R_s & R_s \end{bmatrix} K.
\]  

(12)

\[
G = \begin{bmatrix} 2v & 0 \\ 0 & 2v \left( \frac{R_s}{R_m} \right) \end{bmatrix} \frac{\partial}{\partial X}
\]

\( M, G, K^* \) are the mass, gyroscopic and stiffness operators, respectively, and \( f \) is the vector external forces. In general, \( F_1 \) and \( F_2 \) are functions of \( X \) and \( T \). The equations of motion can be expressed in state space representation as [7]:

\[
\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}_{,,TT} + \begin{bmatrix} 1 & 0 \\ 0 & \left( \frac{R_s}{R_m} \right)^2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}_{,,T} + \begin{bmatrix} (v^2 - \mu^2) & 0 \\ 0 & \left( \frac{R_s}{R_m} \right) v^2 - \left( \frac{R_s}{R_p} \right) \mu^2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}_{,T} + \begin{bmatrix} 1 & -I \\ -R_s & R_s \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\]  

(11)
\[ AU_{1T} + BU = q \]  
where the state and excitation vectors are:
\[ U = \begin{bmatrix} W_{11T} & W_{21T} & W_{1T} & W_{2T} \end{bmatrix}^T, \quad q = \begin{bmatrix} f_1 & f_2 & 0 & 0 \end{bmatrix}^T \]  
and the matrix differential operators are:
\[ A = \begin{bmatrix} M & 0 \\ 0 & K^* \end{bmatrix}; \quad B = \begin{bmatrix} G & K^* \\ -K^* & 0 \end{bmatrix} \]  
Equation (13) is the canonical form of the equation of motion (10), where \( A \) is a symmetric and \( B \) is a skew symmetric matrix operator. Orthogonality of eigenfunctions with respect to each operator is guaranteed in the canonical form, when \( A \) and \( B \) are symmetric and skew symmetric, respectively [7][8][9]. The inner product of two vectors \( U_1 \) and \( U_2 \) is defined as:
\[ \langle U_1, U_2 \rangle = \int_0^l U_1^T \overline{U}_2 dX \]  
where the over bar denotes complex conjugation. The general solution Eq. (13) can be written in the form:
\[ U(X, T) = \text{Re} \left\{ \hat{\phi}_1(X)\overline{\lambda} e^{\overline{\gamma} T} \quad \hat{\phi}_2(X)\overline{\lambda} e^{\overline{\gamma} T} \quad \hat{\phi}_1(X)e^{\overline{\gamma} T} \quad \hat{\phi}_2(X)e^{\overline{\gamma} T} \right\}^T \]  
where the eigenvalues \( \overline{\lambda} (= i\omega) \) are purely imaginary, with \( i = \sqrt{-1} \), and the eigenfunctions \( \hat{\phi} \) are complex.

### 4. NATURAL FREQUENCY ANALYSIS

In order to obtain the natural frequencies and the mode shapes of the system, the homogenous form of equations (5) and (6) are considered. The response of beam-2 is expressed in terms of the response of the beam-1, from equations (5) as follows:
Equations (5) and (6) are then combined into a single eighth-order partial differential equation:

\[
\frac{\partial^8 W_l}{\partial X^8} + A_1 \frac{\partial^8 W_l}{\partial X^8} + A_2 \frac{\partial^8 W_l}{\partial X^8} + A_3 \frac{\partial^8 W_l}{\partial X^8} + A_4 \frac{\partial^8 W_l}{\partial X^8} + A_5 \frac{\partial^8 W_l}{\partial X^8} + A_6 \frac{\partial^8 W_l}{\partial X^8} + A_7 \frac{\partial^8 W_l}{\partial X^8} + A_8 \frac{\partial^8 W_l}{\partial X^8} + A_9 \frac{\partial^8 W_l}{\partial X^8} + A_{10} \frac{\partial^8 W_l}{\partial X^8} + A_{11} \frac{\partial^8 W_l}{\partial X^8} = 0
\]  

(19)

The constant coefficients \( A_n \) \((n = 1\ldots11)\), which are given in Appendix B, depend on the system parameters. Considering the solution given in Eq. (17) in the above equation, the eigenfunction for beam-1 becomes:

\[
\hat{\phi}(X) = \sum_{k=l}^{s} c_k e^{\gamma_k X}
\]  

(20)

where \( c_k \) are constant coefficients, and \( \gamma_k \) are the roots of the characteristic equation of Eq. (19).

This characteristic equation is obtained by substituting \( e^{\gamma_k X} e^{TX} \) into Eq. (19). The roots of this equation are obtained using Mathematica\textsuperscript{TM}, but they are omitted here due to space limitations. The eigenfunction for beam-2 is found by substituting Eq. (20) into Eq. (18), and becomes:

\[
\hat{\phi}(X) = \sum_{k=l}^{s} B_k e^{\gamma_k X}
\]  

(21)

with,

\[
B_k = \frac{1}{K} \left[ \gamma_k^2 + (v^2 - \mu^2) \gamma_k^2 + 2v \gamma_k \lambda + \lambda^2 + K \right].
\]  

(22)

In order to obtain the eigenvalues for the double-beam system, boundary conditions, in Eqs. (8-9) are evaluated using Eqs. (20-21). This results in eight homogeneous algebraic equations, which are represented in matrix form as:
\[ D(\lambda)c = 0 \]  

(23)

where \( c = \{c_1, c_2, c_3, \ldots, c_n\}^T \) is the coefficient vector, and \( D(\lambda) \) is the matrix of coefficients. In order to have a nontrivial solution, the determinant of matrix \( |D(\lambda)| \) must be zero. This gives the characteristic equation of the system. The natural frequencies are determined from the solution of the characteristic equation. A computer program using Mathematica\textsuperscript{TM} is developed to determine these complex natural frequencies. The mode shapes are then calculated from Eqs. (20) and (21) and normalized using the real parts of the complex mode shapes with respect to the symmetric matrix operator \( \mathbf{A} \) as \( \langle A\phi_m^R, \phi_n^R \rangle = \delta_{mn} \).

5. RESULTS AND DISCUSSION

5.1 Mode Shapes and Natural Frequencies

The natural frequencies of the translating double beam system are divided into two fundamental odd and even sets \( \omega_{2n} \) and \( \omega_{2n+1} \), for \( n = 1, 2, \ldots \) The distinction becomes clear in Fig. 2 where the first four mode shapes, corresponding to the first four natural frequencies are plotted, for the two identical beams \( (R_m = R_p = R_s = 1) \) translating with \( v = 5 \), with \( K = 100 \) and \( \mu = 10 \). As stated the mode shapes have real and imaginary parts. Both the real and the imaginary parts of the mode shapes for \( \omega_1 \) and \( \omega_3 \) show that the two beams experience in-phase deflections. On the other hand, the mode shapes for \( \omega_2 \) and \( \omega_4 \) show out-of-phase deflections. It is also seen that the mode shapes are not symmetrical with respect to the mid-span of the beams, due to the effects of translation, as was also observed by Wickert and Mote [9]. The in-phase mode shapes of the two beams are anti-symmetric with respect to the mid-height of the thickness of the two beams.
beam system, while the out-of-phase mode shapes are symmetric. Thus, the in-phase modes of identical beams do not deflect (i.e., stretch/compress) the elastic foundation; while opposite is true for the out-of-phase modes. The presence of in-phase and out-of-phase modes is the result of the coupling of the two beams by the Winkler foundation, and it is observed for non-translating beam systems (e.g., [19][25]).

In case the beams are not identical, the vibration of the two beams still show in-phase and out-of-phase characteristics, for odd- and even-modes, respectively, as shown in Fig. 3, for a case where the mass ratio $R_m = 0.6$. Close inspection of the figure shows that the mode shapes of the two beams are not parallel to each other. This effect is observed for other $R_m$ values, as well as $R_p$ and $R_s$ values, and parallelism of the modes deteriorate further with decreasing values of $R_m$, $R_p$, and $R_s$. Moreover, the symmetry and anti-symmetry of the in-phase and out-of-phase mode shapes, respectively, are distorted as the material properties are not identical. For non-identical traveling beams, both of the in-phase and the out-of-phase modes are always affected by the coupling between the two beams.

5.2 Effect of Translation Speed on Natural Frequencies

The effect of the non-dimensional translation speed $\nu$ on the natural frequencies of the two identical beams described above are shown in Fig. 4. Similar to a single, tensioned, simply supported, axially moving beam analyzed in reference [9], the first natural frequency vanishes at the critical speed $\nu_c = \left(\mu^2 + \pi^2\right)^{1/2}$. The eigenvalues for the double beam system are purely imaginary for $\nu < \nu_c$, but become complex for larger values of $\nu$. The first natural frequency is not affected by the presence of the second beam. This result is expected, as the odd numbered
natural frequencies (in-phase) are not affected by the presence of the Winkler foundation for identical beams. Hence, the onset of divergence instability for the double beam system analyzed here is identical to the case of the single beam, and the elastic stiffness does not alter the divergence instability. The even numbered frequencies behave in a similar way to the odd frequencies, except their divergence occurs at larger translation speeds due to the coupling for between the beams through the elastic foundation.

The real part of the natural frequencies plotted in Fig. 4b, shows that flutter instabilities occur along with divergence for translation speeds greater than $v_c$. This behavior is not altered as compared to the single beam case. Moreover, the out-of-phase frequency spectrum behaves qualitatively the same way as the in-phase part.

5.3 Effect of Foundation Stiffness on Natural Frequencies

In order to demonstrate the effect the elastic foundation stiffness $K$ on the vibration of the system, the first six natural frequencies of the system are investigated for different values of the non-dimensional tension ratio $R_p$, mass ratio $R_m$, and bending rigidity ratio $R_s$, in Fig. 5. The natural frequencies are investigated for $0 \leq K \leq 500$, while the base values are chosen as $R_m = R_s = R_p = 1, \mu = 10$, and $\nu = 5$.

The effect of varying the elastic foundation stiffness $K$ for two identical beams is presented in Figure 5-a. This figure shows that the frequencies of the in-phase modes ($\omega_1, \omega_5, \omega_7$) are not affected by the stiffness of the foundation, $K$, and remain 53, 131 and 242, respectively, as expected. On the other hand, the out-of-phase mode frequencies,($\omega_2, \omega_6, \omega_8$) increase slightly with increasing $K$. This increase is due to the general stiffening of the system with increasing $K$ values.
The effect of the mass ratio parameter $R_m = 0.1$ is shown in Fig. 5-b. Considering the normalization described in Eq. (7), the value of $R_m = 0.1$ should be thought as a ten fold increase in the mass of one of the beams (e.g., beam-2), while all of the other non-dimensional parameters affected by mass of this beam remain constant. This figure shows that decreasing the mass ratio to $R_m = 0.1$ causes the natural frequencies to increase slightly with respect to Fig. 5-a. The in-phase frequencies ($\omega_1$, $\omega_3$, $\omega_5$) are no-longer independent of the stiffness $K$, showing that the beams are coupled. In particular, $\omega_5$ is seen to become lower with respect to $R_m = 1$ case. But, otherwise the natural frequencies remain nearly constant with increasing $K$.

The effect of bending rigidity ratio $R_s = 0.1$ is given in Fig. 5-c. Considering a ten fold difference in bending rigidity between the two beams, it is seen that the in-phase modes ($\omega_1$, $\omega_3$, $\omega_5$) are affected slightly with increasing $K$ values; this points out that the beams are coupled through the elastic foundation when the bending rigidities are not identical. When $R_s = 0.1$ the out-of-phase mode frequencies ($\omega_2$, $\omega_4$, $\omega_6$) move to higher values. It is clear that increasing the elastic foundation stiffness $K$ does not significantly affect the natural frequencies.

The effect of non-identical tension distribution between the beams, for different $K$ values is shown for $R_p = 0.3$ in Fig. 5-d. Similar to the previous two non-identical cases just considered the in-phase frequencies ($\omega_1$, $\omega_3$, $\omega_5$) become slightly dependent on $K$; and the out-of-phase modes frequencies become higher with $R_p = 0.3$. As in the other cases the elastic foundation stiffness has a small effect on the frequencies.

In summary, rendering $R_m$, $R_s$, and $R_p$ values less than one makes the in-phase frequencies become slightly dependent on $K$; and, increases the out-of-phase frequencies. This effect is expected as mass, bending-rigidity and tension of one of the beams increase with decreasing $R_m$, $R_s$, and $R_p$ values.
5.4 Effect of Axial Tension on Natural Frequencies

Next, the effects the non-dimensional tension parameter $\mu$ on the natural frequencies of the system are investigated in the range $0 \leq \mu \leq 10$. The results are shown in Fig. 6, where the natural frequencies are presented for different values of $R_m$, $R_s$, and $R_p$ for the subcritical translation velocity of $\nu = 5$. As before, the base values are chosen for two identical beams, as $R_m = R_s = R_p = 1$, $K = 100$, unless otherwise noted.

The effect of applying different non-dimensional tension ($\mu$) values on the natural frequencies is shown in Figure 6-a, for the case of identical beams. This figure shows that frequencies increase with increasing $\mu$, while the values of the natural frequencies for the in-phase and out-of-phase modes remain relatively close each other. An exception occurs near $\mu_{cr} = 3.9$, where the critical non-dimensional tension ($\mu_{cr}$), defined here as the $\mu$ value where first natural frequency of the coupled system becomes zero. This $\mu_{cr}$ value is valid for non-dimensional speed value of $\nu = 5$.

The effect of non-identical beam mass is investigated by letting the mass ratio $R_m = 0.1$. Results are presented in Fig. 6-b, which shows that the frequencies of the in-phase modes ($\omega_1$, $\omega_3$, $\omega_5$) increase only slightly; while, the frequencies of the out-of-phase modes ($\omega_2$, $\omega_4$, $\omega_6$) move to higher values with respect to the identical beam case given in Fig. 6-a. The critical value of the tension parameter becomes $\mu_{cr} \cong 3.2$.

The effect of bending rigidity ratio, $R_s = 0.1$, is shown in Fig. 6-c. This figure shows that the first two in-phase-mode frequencies ($\omega_1$, $\omega_3$) are not affected significantly by the reduction in the non-dimensional bending rigidity $R_s$, however the third in-phase-mode frequency ($\omega_5$) and the
out-of-phase-mode frequencies increase, with respect to case of identical beams. The critical value of the tension parameter is approximately $\mu_{cr} \cong 2$.

The effect of non-equal tension distribution between the two beams is shown in Fig. 6-d, where the axial load ratio $R_p = 0.3$. The in-phase-mode frequencies are not affected significantly while the out-of-phase mode frequencies show a large increase for the higher $\mu$ values with respect to the identical beam case (Fig 6-a). The critical value of the tension parameter is approximately $\mu_{cr} \cong 2.8$.

In summary, by increasing the non-dimensional tension $\mu$, the overall stiffness of the system increases, causing the natural frequencies of the in-phase and out-of-phase-modes to increase. Increasing the mass, bending-rigidity and tension of one of the beams, as described in Section 5.3, affects the out-of-phase-modes; and causes their natural frequencies to increase. Out-of-phase modes are affected more strongly by increasing tension, as the system represented by these modes is more strongly subjected to the effect of the foundation stiffness.

5.5 Effect of Axial Tension on Critical Speed and Stability

As discussed in Section 5.2, the critical speed $\nu_c$ depends on the axial tension. Stability regions as a function of non-dimensional translation speed $\nu$ and non-dimensional tension $\mu$ are presented next. Figure 7-a shows the effect of different mass ratios ($R_m = 1, 0.9, 0.6, 0.3$) on the stability of the system for $K = 100$, $R_s = R_p = 1$. It is seen that lowering the mass ratio $R_m$ reduces the stable domain. This is expected, as lowering the mass ratio causes the mass of one of the beams to increase. On the other hand, the bending rigidity $R_s$, and axial tension $R_p$ ratios have a small effect on the stability regions as shown in Figs. 7-b and 7-c. Fig. 7 also shows that the non-
The free transverse vibration of an elastically connected axially loaded, simply supported, axially translating double beam system is considered. The two beams have the same length, translation speed, and boundary conditions. The system of governing partial differential equations is cast in the first order canonical form as state space form. The natural frequencies and mode shapes are obtained. In general, the natural frequencies of the system are composed of two infinite sets, $\omega_n$ and $\omega_{2n}$. When the two beams are identical, the free vibrations are described by synchronous and asynchronous vibrations, with $\omega_n$ and $\omega_{2n}$, respectively. The vibrations still show in-phase and out-of-phase characteristics, as the parameters of the beams change. The effects of beams properties, elastic foundation stiffness, translating velocity, and axial tension are investigated. Divergence instability occurs at the critical speed, and flutter and divergence instabilities coexist in post critical speeds. It is found that, in the case of identical beams the presence of the elastic foundation does not affect the critical speed. The stability regions are obtained as a function of axial tension and corresponding critical speed for different mass, bending rigidity, and axial loading ratios, and found that the variation of mass ratio has significant effect on critical speeds.

6. SUMMARY AND CONCLUSIONS

The free transverse vibration of an elastically connected axially loaded, simply supported, axially translating double beam system is considered. The two beams have the same length, translation speed, and boundary conditions. The system of governing partial differential equations is cast in the first order canonical form as state space form. The natural frequencies and mode shapes are obtained. In general, the natural frequencies of the system are composed of two infinite sets, $\omega_n$ and $\omega_{2n}$. When the two beams are identical, the free vibrations are described by synchronous and asynchronous vibrations, with $\omega_n$ and $\omega_{2n}$, respectively. The vibrations still show in-phase and out-of-phase characteristics, as the parameters of the beams change. The effects of beams properties, elastic foundation stiffness, translating velocity, and axial tension are investigated. Divergence instability occurs at the critical speed, and flutter and divergence instabilities coexist in post critical speeds. It is found that, in the case of identical beams the presence of the elastic foundation does not affect the critical speed. The stability regions are obtained as a function of axial tension and corresponding critical speed for different mass, bending rigidity, and axial loading ratios, and found that the variation of mass ratio has significant effect on critical speeds.
REFERENCES


APPENDIX A: FOUNDATION STIFFNESS MODEL

Adhesion due to liquid films in the contact interface can cause a substantial increase in the normal force required to separate the surfaces. Depending on the amount of liquid in the interface (Fig. A.1) the meniscus force between a sphere and a flat surface can be given by one of the following relations:

\[ F^c = 2\pi R \gamma_1 (\cos \theta_1 + \cos \theta_2), \]  
\[ F^{nc} = \frac{2\pi R \gamma_1 (\cos \theta_1 + \cos \theta_2)}{\left(1 + \frac{D}{d}\right)}, \]  
\[ F^h = 2\pi R \gamma_1 (1 + \cos \theta_1), \]  

where \( F^c \) is the meniscus force in case of contact (Fig. A.1a) [26]; \( F^{nc} \) is the meniscus force between two non-contacting asperities (Fig. A.1b) [26]; \( F^h \) is the meniscus force when an asperity is dipped in a pool of liquid with height \( h \) (Fig. A.1c) [27][28]. The contact angle between the liquid and the asperity is \( \theta_1 \) and between the liquid and the surface is \( \theta_2 \), \( R \) is the radius of sphere, \( \gamma_1 \) is the surface tension of the liquid, \( D \) is the separation of the two surfaces, \( d = s - D \), and \( s \) is the meniscus height. Note that the adhesion force in cases-1 and -3 are independent of \( D \). For all three cases, it is reasonable to assume that the meniscus force immediately vanishes at a critical value of clearance \( D_{cr} \), during pull-off.

Figure A.2 shows a simple model of the interactions of the liquid menisci with the surfaces of two beams in close proximity. While the peak-height variation of surface roughness is Gaussian for most engineering surfaces, here we represent the asperities as large spheres on one surface while the other surface is flat. This assumption enables easy visualization of the capillary-adhesion. The three regimes of capillary action are shown in Figs. A.2a-A.2c. In
practice, combination of these cases would be present at an interface. The governing equation for
the two beam system with the effect of capillary forces $F_c$ can be given in general as:

$$E_1 j_1 \frac{d^4 w_1}{dx^4} - T_1 \frac{d^2 w_1}{dx^2} + F_c + m_1 \left( \frac{\partial^2 w_1}{\partial t^2} + 2\nu \frac{\partial^2 w_1}{\partial x \partial t} + \nu^2 \frac{\partial^2 w_1}{\partial x^2} \right) = f_1(x,t) \tag{A.4}$$

$$E_2 j_2 \frac{d^4 w_2}{dx^4} - T_2 \frac{d^2 w_2}{dx^2} - F_c + m_2 \left( \frac{\partial^2 w_2}{\partial t^2} + 2\nu \frac{\partial^2 w_2}{\partial x \partial t} + \nu^2 \frac{\partial^2 w_2}{\partial x^2} \right) = f_2(x,t) \tag{A.5}$$

where $F_c$ is one of $F^c$, $F^{nc}$ or $F^h$. Only $F^{nc}$ depends on separation, $D = w_1 - w_2$, and $F^c$ and
$F^h$ are constant adhesion forces. Ordinarily, in vibration analysis the constant adhesion forces
would have to be moved to the right hand side, and their effect would disappear from modal
analysis. In order to keep the effect of the attractive capillary force, we replace $F_c$ with
$k(w_1 - w_2)$ where the spring stiffness $k$ is determined from work equivalency of the adhesion
forces and the simple spring model. For example, in the case of $F^c$, $k$ would become:

$$F^c D_{cr1} = \frac{1}{2} k_1 D_{cr1}^2$$

giving

$$k_1 = 2 \frac{F^c}{D_{cr1}} = 2\pi R \gamma (\cos \theta_1 + \cos \theta_2) \div D_{cr1} \tag{A.6}$$

and similarly for $F^{nc}$ and $F^h$, we get, respectively:

$$k_2 = \frac{1}{D_{cr2}^2} \int_0^{D_{cr2}} 2F^{nc} dD = \frac{2}{D_{cr2}^2} \left( \frac{1}{2} D_{cr2} \left( 2\pi R \gamma (\cos \theta_1 + \cos \theta_2) - F_0 \right) \right) + D_{cr2} F_0 \tag{A.7}$$

and
Consider a section of a 2 m wide, 100 µm thick web, spanning 6 m between two supports, with 10 GPa elastic modulus, and 70 mJ/ m² surface tension. The contact area can be modeled as asperities on a flat surface as depicted in Fig. A.2, with 0.001 m asperity radius and 300 asperities/m². Under the case-1 contact assumption, the adhesion force for single asperity can be calculated using Eq. (A.6) as $F^c = 4.4 \times 10^{-4}$ N. The total adhesion force becomes $F^c_{total} = 4.4 \times 10^{-4}(300)(12) = 1.58$ N. For an assumed snap-off distance of $D_{cr} = 1 \mu$m, the elastic foundation stiffness from Eq. (A.6) becomes $k_l = 3.17 \times 10^6$ N/m. With this value, the non-dimensional elastic foundation stiffness $K$ defined in Eq. (7) becomes $K = 2.5 \times 10^{12}$.

As a point of reference the elastic foundation stiffness $k_c$ for an adhesive layer (3M ISD-110) is calculated from reference [21] at room temperature ($T = 70$ degrees Fahrenheit) for a frequency of ($F = 50$ Hz), and assuming Poisson’s ratio $v_c = 0.5$. The elastic modulus of the adhesive under these conditions is found from Table 1 and Eqs. (19) and (22) of this reference as $E_c = 0.366$ MPa. Using the relation $k = E_c/\left((1-v_c^2)h_c\right)$ [21], and assuming an adhesive thickness of $h_c = 1$ mm, one finds $k_c = 488\times10^6$ N/m, and the non-dimensional stiffness becomes $K = 3.81\times10^{14}$. Thus one can see that the elastic foundation stiffness predicted from capillary adhesion is approximately two orders of magnitude smaller than an industrial adhesive layer.
APPENDIX B

Coefficients of equation (19) are:

\[ A_i = (1 + \frac{R_s}{R_m})v^2 - (1 + \frac{R_s}{R_p})\mu^2 \]  
(a)

\[ A_2 = 2\nu(1 + \frac{R_s}{R_m}) \]  
(b)

\[ A_3 = (1 + \frac{R_s}{R_m}) \]  
(c)

\[ A_4 = K(1 + R_s) + \left(\frac{R_s}{R_m}\right)v^2 - \frac{R_s}{R_p}\mu^2)(v^2 - \mu^2) \]  
(d)

\[ A_5 = 2\nu\left(\frac{R_s}{R_m}\right)v^2 - \frac{R_s}{R_p}\mu^2 + 2\nu\left(\frac{R_s}{R_m}\right)(v^2 - \mu^2) \]  
(e)

\[ A_6 = \left(\frac{R_s}{R_m}\right)v^2 - \frac{R_s}{R_p}\mu^2 + \left(\frac{R_s}{R_m}\right)(v^2 - \mu^2) + (2\nu)^2\left(\frac{R_s}{R_m}\right) \]  
(f)  
(B.1)

\[ A_7 = 4\nu\left(\frac{R_s}{R_m}\right) \]  
(g)

\[ A_8 = \left(\frac{R_s}{R_m}\right) \]  
(h)

\[ A_9 = K\left(\frac{R_s}{R_m}\right)v^2 - \frac{R_s}{R_p}\mu^2 + R_sK(v^2 - \mu^2) \]  
(i)

\[ A_{10} = R_sK\left(1 + \frac{1}{R_m}\right) \]  
(j)

\[ A_{11} = 2\nu \cdot R_sK\left(1 + \frac{1}{R_m}\right) \]  
(k)
LIST OF FIGURES

**Figure 1** Double beams connected by elastic foundation. The first four complex mode shapes. The real (solid) and imaginary parts (dashed) for $K = 100, \nu = 5, \mu = 10, R_m = R_p = R_s = 1$.

**Figure 2** The first four complex mode shapes for two identical beams traveling with non-dimensional speed of $\nu = 5$. The real (solid) and imaginary parts (dashed) for $K = 100, \mu = 10, R_p = R_s = R_m = 1$.

**Figure 3** The first four complex mode shapes for a traveling system with non-equal mass densities ($R_m = 0.60$). The real (solid) and imaginary parts (dashed) for $K = 100, \nu = 5, \mu = 10, R_p = R_s = 1$.

**Figure 4** The imaginary and real parts of the frequency spectrum for translating double beams system; the solid curves indicate the synchronous frequencies and dashed curves indicate the asynchronous frequencies. ($K = 100, \mu = 10, R_m = R_s = R_p = 1$).

**Figure 5** The natural frequencies versus non-dimensional elastic foundation stiffness $K$ for $\mu = 10, \nu = 5$.

**Figure 6** The natural frequencies versus non-dimensional axial tension parameter, $\mu$, for $K = 100, \nu = 5$.

**Figure 7** The non-dimensional transport speed $\nu$ versus non-dimensional axial tension parameter, $\mu$, for $K = 100, R_s = R_p = 1$.

**Figure A.1** Schematic depiction of menisci between a flat surface and a sphere for the cases of a) contact, b) no-contact with minimal coverage and c) no-contact with liquid flooded interface.

**Figure A.2** A schematic of the uniform-spherical asperities on flat surface.
Figure 1
Figure 2
Figure 3

a) $\omega_1$

b) $\omega_2$

c) $\omega_3$

d) $\omega_4$
Figure 4
Figure 5
Figure 6

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\begin{align*}
\text{Im}(\lambda_n) & \quad \text{Axial Tension, } \mu \\
0 & \quad 50 & \quad 100 & \quad 150 & \quad 200 & \quad 250 \\
0 & \quad 50 & \quad 100 & \quad 150 & \quad 200 & \quad 250 \\
0 & \quad 50 & \quad 100 & \quad 150 & \quad 200 & \quad 250 \\
0 & \quad 50 & \quad 100 & \quad 150 & \quad 200 & \quad 250 \\
\omega_1 & \quad \omega_2 & \quad \omega_3 & \quad \omega_4 & \quad \omega_5 & \quad \omega_6 \\
\end{align*}
```
Figure 7

a) $K = 100, R_s = R_p = 1$

b) $K = 100, R_m = R_p = 1$

c) $K = 100, R_s = R_m = 1$

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Figure A.1

Figure A.2